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Wave propagation in RTD-based cellular neural networks

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Abstract

This work investigates the existence of monotonic traveling wave and standing wave solutions of RTD-based cellular neural networks in the one-dimensional integer lattice $\mathbf{Z}^1$. For nonzero wave speed c, applying the monotone iteration method with the aid of real roots of the corresponding characteristic function of the profile equation, we can partition the parameter space (γ, δ)-plane into four regions such that all the admissible monotonic traveling wave solutions connecting two neighboring equilibria can be classified completely. For the case of $c = 0$, a discrete version of the monotone iteration scheme is established for proving the existence of monotonic standing wave solutions. Furthermore, if γ or δ is zero then the profile equation for the standing waves can be viewed as an one-dimensional iteration map and we then prove the multiplicity results of monotonic standing waves by using the techniques of dynamical systems for maps. Some numerical results of the monotone iteration scheme for traveling wave solutions are also presented.

MSC: 34B15; 34B45; 34K10; 34K28

Keywords: Lattice dynamical systems; RTD-based cellular neural networks; Discrete Fisher equation; Discrete Nagumo equation; Monotone iteration methods; Traveling waves; Standing waves

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1. Introduction

In this paper, we establish the existence of monotonic traveling wave and standing wave solutions of RTD-based cellular neural networks (CNNs) by using the techniques of monotone iteration coupled with the concept of upper and lower solutions in the theory of monotone dynamical systems.

This study is motivated by the recent work of Itoh et al. [12] in which they reported that the resonant tunneling diode (RTD), a class of quantum effect devices, is an excellent candidate for both analog and digital nanoelectronics applications because of its structural simplicity, relative ease of fabrication, inherent high speed and design flexibility. The dynamics of the two-dimensional RTD-based CNN with a neighborhood of radius r are governed by a system of $n = MN$ differential equations

$$\frac{dx_{ij}(t)}{dt} = -g(x_{ij}(t)) + \sum_{k,\ell \in N_{ij}} (a_{k-i,\ell-j}x_{k\ell}(t) + b_{k-i,\ell-j}u_{k\ell}(t)) + z_{ij}, \tag{1.1}$$

for $(i,j) \in \{1, 2, \dots, M\} \times \{1, 2, \dots, N\}$ and $t \in \mathbf{R}$, where N_{ij} denotes the r-neighborhood of cell C_{ij}; $a_{k\ell}$, $b_{k\ell}$ and z_{ij} denote the feedback, control and threshold template parameters, respectively; $x_{k\ell}(t)$ and $u_{k\ell}(t)$ denote the state and input functions of cell $C_{k\ell}$, respectively. For the sake of simplicity, the v-i characteristic of the RTD is modeled by a piecewise-linear function g which is given by

$$\begin{aligned} g(x_{ij}) &= \alpha x_{ij} + \beta(|x_{ij} - V_p| - |x_{ij} - V_v|) - \beta(|x_{ij} + V_p| - |x_{ij} + V_v|) \\ &= \begin{cases} \alpha x_{ij} + m + \beta(|x_{ij} - V_p| - |x_{ij} - V_v|) & \text{if } x_{ij} \geqslant 0, \\ \alpha x_{ij} - m - \beta(|x_{ij} + V_p| - |x_{ij} + V_v|) & \text{if } x_{ij} \leqslant 0, \end{cases} \end{aligned} \tag{1.2}$$

where $m = \beta(V_v - V_p)$, $\alpha > 0$, $\beta < 0$ are constants, and V_p, V_v $(0 < V_p < V_v)$ are the peak and valley voltages of the RTD for the positive region of x_{ij}, respectively. Notice that the function g is symmetric with respect to the origin as shown in Fig. 1.

The use of RTDs in applications to cellular neural networks has been previously reported in [2,5]. A circuit implementation of the RTD-based CNN can be found in [12]. It is also pointed out in [12] that the bistable RTD-based CNN exhibits good performance for a number of interesting image processing applications because of its high-speed processing and high cell density. Thus, it is possible that a new generation of low-power, high-speed, and large array-size CNNs appears with the introduction of the RTD-based CNN. Many methods used in image processing and pattern recognition can be easily implemented by the RTD-based CNN approach, however, the mathematical analyses of the pattern formation, spatial chaos properties, and its dynamical behavior are still not fully documented (see [2,5,9,12]).

In this article, we are interested in studying the existence of monotonic traveling wave and standing wave solutions of the one-dimensional original RTD-based CNN

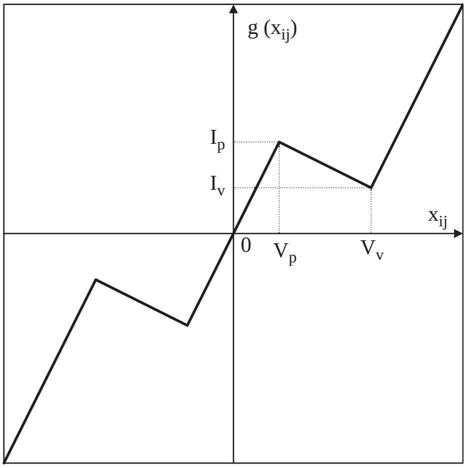


Fig. 1. The *v-i* characteristic of the RTD.

without input and threshold terms defined as follows:

$$\frac{dx_i(t)}{dt} = -g(x_i(t)) + ax_i(t) + \gamma x_{i-1}(t) + \delta x_{i+1}(t), \tag{1.3}$$

for all $i \in \mathbf{Z}^1$ and $t \in \mathbf{R}$, where the real parameters a, γ, δ with $\gamma \geqslant 0$, $\delta \geqslant 0$, and $\gamma + \delta \neq 0$ constitute the so-called space-invariant template that measures the synaptic weights of self-feedback and neighborhood interaction.

A continuously differentiable function $\varphi: \mathbf{R} \to \mathbf{R}$ such that

$$x_i(t) = \varphi(i - ct), \quad \text{for all } i \in \mathbf{Z}^1 \text{ and } t \in \mathbf{R} \tag{1.4}$$

is a solution of system (1.3) is called a traveling wave solution, where $s := i - ct \in \mathbf{R}$ is the moving coordinate for a given nonzero wave speed $c \in \mathbf{R}$. Thus, under assumption (1.4), the profile equation of system (1.3) can be written as

$$-c\varphi'(s) = -g(\varphi(s)) + a\varphi(s) + \gamma\varphi(s-1) + \delta\varphi(s+1), \tag{1.5}$$

and we are attempting to prove the existence of monotonic solutions of (1.5) supplemented with some appropriate asymptotic boundary conditions that will be specified later. On the other hand, if $c = 0$ then the propagation of traveling profile fails and we are seeking a discrete monotonic profile function $\varphi: \mathbf{Z} \to \mathbf{R}$ satisfying

$$0 = -g(\varphi(i)) + a\varphi(i) + \gamma\varphi(i-1) + \delta\varphi(i+1). \tag{1.6}$$

Such a solution $\{\varphi(i)\}_{i \in \mathbf{Z}}$ of (1.6) is called a monotonic standing wave solution of system (1.3).

The study of traveling wave and standing wave solutions for partial differential equations and lattice dynamical systems has drawn considerable attention in the past decades. The existence and stability of such solutions has been much studied for lattice dynamical systems (see, e.g., [1,3,4,6–11,13–21] and many references therein). For example, if $a = -2d$, $\gamma = \delta = d > 0$, and $g(u) = u(u-1)$ or $g(u) = u(u-k)(u-1)$ for some $0 < k < 1/2$, the equations are, respectively, called the discrete Fisher equation and discrete Nagumo equation. These typical equations possessing traveling wave or standing wave solutions have wide applications in various fields, from chemistry and biology to physics and engineering. We refer the reader to [1,4,14,19,20] for more details, see also Section 2.5.

For Eqs. (1.5) and (1.6), we are interested in finding monotonic profile solutions satisfying some specific boundary conditions, namely, the heteroclinic orbits of (1.5) and (1.6) connecting two neighboring equilibria. To guarantee the existence of equilibria of (1.5) and (1.6), some conditions for the parameters in (1.3) are required as following:

$$0 \leqslant \alpha + \frac{2m}{V_v} < a + \gamma + \delta < \alpha \leqslant -2\beta. \tag{1.7}$$

If (1.7) holds, then a simple computation shows that there are five homogeneous solutions x_0, $x_1^{\pm}$, and $x_2^{\pm}$ of system (1.3) given by (cf. Fig. 2)

$$x_0 = 0, \quad x_1^{\pm} = \pm \frac{-2\beta V_p}{a + \gamma + \delta - \alpha - 2\beta}, \quad \text{and} \quad x_2^{\pm} = \pm \frac{2\beta(V_v - V_p)}{a + \gamma + \delta - \alpha}. \tag{1.8}$$

Therefore, in the following of this article we always assume that (1.7) holds. We are mainly interested in finding the monotonic traveling wave and standing wave solutions of system (1.3) that satisfy the following asymptotic boundary conditions:

$$(\mathrm{BC1}): \quad \lim_{s \to -\infty} \varphi(s) = x_0 \quad \text{and} \quad \lim_{s \to \infty} \varphi(s) = x_1^{+},$$

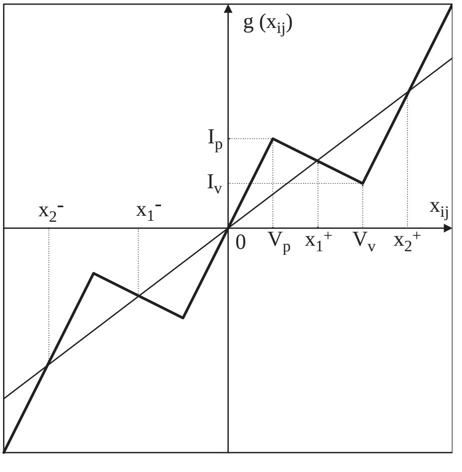


Fig. 2. The five homogeneous equilibria of (1.3).

$$(\mathrm{BC2}): \quad \lim_{s\to-\infty}\varphi(s)=x_1^+ \quad \text{and} \quad \lim_{s\to\infty}\varphi(s)=x_0,$$

$$(\mathrm{BC3}): \quad \lim_{s\to-\infty}\varphi(s)=x_1^+ \quad \text{and} \quad \lim_{s\to\infty}\varphi(s)=x_2^+,$$

$$(\mathrm{BC4}): \quad \lim_{s\to-\infty}\varphi(s)=x_2^+ \quad \text{and} \quad \lim_{s\to\infty}\varphi(s)=x_1^+.$$

It is to be understood that $\lim_{s\to\pm\infty}\varphi(s)$ means $\lim_{i\to\pm\infty}\varphi(i)$ for considering standing wave solutions.

Using the monotone iteration scheme, the authors in [6] considered more general setting of functional differential equations than (1.5) and proved the existence of monotonic traveling wave solutions. Unfortunately, their results cannot be applied directly to our Eq. (1.5) with the above-mentioned various asymptotic boundary conditions. However, in this paper, we will still apply the similar monotone iteration techniques with the concept upper and lower solutions to obtain the monotonic traveling wave solutions, and to classify all the monotonic traveling wave solutions for the given various parameters. To this aim, we first partition the nonnegative (γ,δ)-plane into four regions Ω_i, $i=1,2,3,4$ (cf. Fig. 3),

$$\Omega_1=\{(\gamma,\delta)\,|\,0\leqslant\gamma<\delta \text{ and } 2\sqrt{\gamma\delta}+a<\alpha+2\beta\},$$

$$\Omega_2=\{(\gamma,\delta)\,|\,0\leqslant\delta<\gamma \text{ and } 2\sqrt{\gamma\delta}+a<\alpha+2\beta\},$$

$$\Omega_3=\{(\gamma,\delta)\,|\,0<\gamma\leqslant\delta \text{ and } 2\sqrt{\gamma\delta}+a\geqslant\alpha+2\beta\},$$

$$\Omega_4=\{(\gamma,\delta)\,|\,0<\delta<\gamma \text{ and } 2\sqrt{\gamma\delta}+a\geqslant\alpha+2\beta\}.$$

We remark that if $a\geqslant\alpha+2\beta$ then Ω_1 and Ω_2 vanish and, in this case, we allow $\gamma=0$ and $\delta=0$ in Ω_3 and Ω_4, respectively. Notice that, in Ω_1 and Ω_2, the constraint

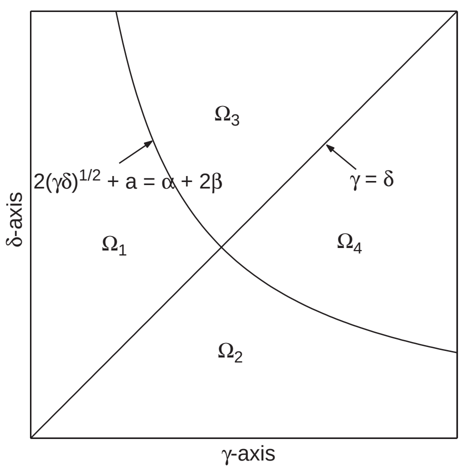


Fig. 3. Partition of the nonnegative (γ,δ)-plane.

$2\sqrt{\gamma\delta} + a < \alpha + 2\beta$ means somewhat the coupling strength is small in the coupled system (1.3), while in Ω_3 and Ω_4, the coupling strength is large enough. In each region Ω_i, by studying the properties of the corresponding characteristic function of (1.5) at x_1^+, we can completely classify all the monotonic traveling wave solutions of (1.3) with various asymptotic boundary conditions. Indeed, we have the following results:

Theorem 1. *Assume that* (1.7) *holds. Consider the profile equation* (1.5) *with* $c \neq 0$, *then there exist critical speeds* c_i *for* $i = 1, 2, \dots, 20$ *with* $c_{2j-1} = c_{2j}$ *for* $j = 1, 2, \dots, 10$ *such that*

(1) *If* $(\gamma, \delta) \in \Omega_1$ *then there are monotonic traveling wave solutions of* (1.3) *satisfying different boundary conditions as shown in* Table 1.
(2) *If* $(\gamma, \delta) \in \Omega_2$ *then there are monotonic traveling wave solutions of* (1.3) *satisfying different boundary conditions as shown in* Table 2.
(3) *If* $(\gamma, \delta) \in \Omega_3$ *then there are monotonic traveling wave solutions of* (1.3) *satisfying different boundary conditions as shown in* Table 3.
(4) *If* $(\gamma, \delta) \in \Omega_4$ *then there are monotonic traveling wave solutions of* (1.3) *satisfying different boundary conditions as shown in* Table 4.

In Theorem 1, we only consider monotonic solutions of the profile equation (1.5) connecting two neighboring nonnegative equilibria. However, since g is an odd function, if we let $\tilde{\varphi}(s) = -\varphi(s)$ for $s \in \mathbf{R}$ then the profile equation (1.5) is changed into

$$-c\tilde{\varphi}'(s) = -g(\tilde{\varphi}(s)) + a\tilde{\varphi}(s) + \gamma\tilde{\varphi}(s-1) + \delta\tilde{\varphi}(s+1), \tag{1.9}$$

which is exactly the same form as (1.5). Hence, according to Theorem 1, we also obtain the existence of monotonic traveling wave solutions of (1.3) connecting two neighboring nonpositive equilibrium solutions.

Now, making an observation on cases (1-1) and (1-5), we immediately conclude that for the same template (a, γ, δ) there are simultaneously monotonic traveling

Table 1
Traveling wave solutions for $(\gamma, \delta) \in \Omega_1$

Case	Range of speed	Monotonicity of $\varphi(s)$	Boundary condition
(1-1)	$c_1 < c < 0$	Nondecreasing	(BC1)
(1-2)	$c_2 < c < 0$	Nonincreasing	(BC4)
(1-3)	$c < c_3 < 0$	Nonincreasing	(BC2)
(1-4)	$c < c_4 < 0$	Nondecreasing	(BC3)
(1-5)	$c > c_5 = 0$	Nondecreasing	(BC1)
(1-6)	$c > c_6 = 0$	Nonincreasing	(BC4)

Table 2
Traveling wave solutions for $(\gamma, \delta) \in \Omega_2$

Case	Range of speed	Monotonicity of $\varphi(s)$	Boundary condition
(2-1)	$0 < c < c_7$	Nonincreasing	(BC2)
(2-2)	$0 < c < c_8$	Nondecreasing	(BC3)
(2-3)	$0 < c_9 < c$	Nondecreasing	(BC1)
(2-4)	$0 < c_{10} < c$	Nonincreasing	(BC4)
(2-5)	$c < c_{11} = 0$	Nonincreasing	(BC2)
(2-6)	$c < c_{12} = 0$	Nondecreasing	(BC3)

Table 3
Traveling wave solutions for $(\gamma, \delta) \in \Omega_3$

Case	Range of speed	Monotonicity of $\varphi(s)$	Boundary condition
(3-1)	$c > c_{13} > 0$	Nondecreasing	(BC1)
(3-2)	$c > c_{14} > 0$	Nonincreasing	(BC4)
(3-3)	$c < c_{15} < 0$	Nonincreasing	(BC2)
(3-4)	$c < c_{16} < 0$	Nondecreasing	(BC3)

Table 4
Traveling wave solutions for $(\gamma, \delta) \in \Omega_4$

Case	Range of speed	Monotonicity of $\varphi(s)$	Boundary condition
(4-1)	$c < c_{17} < 0$	Nonincreasing	(BC2)
(4-2)	$c < c_{18} < 0$	Nondecreasing	(BC3)
(4-3)	$c > c_{19} > 0$	Nondecreasing	(BC1)
(4-4)	$c > c_{20} > 0$	Nonincreasing	(BC4)

wave solutions which satisfy (BC1) with positive and negative wave speeds. Similar situations occur in cases (1-2) and (1-6), cases (2-1) and (2-5), and case (2-2) with (2-6). Therefore, it is natural to ask what happens when the wave speed c is zero. If $c = 0$ then the propagation of the traveling profile fails. In this case, by using a discrete monotone iteration method, we still can obtain the monotonic standing wave solutions with various asymptotic boundary conditions. More specifically, if $\gamma = 0$ or $\delta = 0$ then Eq. (1.6) can be viewed as an one-dimensional iteration map, and we can prove the multiplicity results of monotonic standing waves by the techniques of dynamical systems for maps. The precise statements of the results are:

Theorem 2. *Assume that* (1.7) *holds. Consider the profile equation* (1.6), *then we have*

(1) *If* $(\gamma, \delta) \in \Omega_1$, *then there are monotonic nondecreasing and nonincreasing standing wave solutions of* (1.3) *satisfying* (BC1) *and* (BC4), *respectively.*
(2) *If* $(\gamma, \delta) \in \Omega_2$, *then there are monotonic nonincreasing and nondecreasing standing wave solutions of* (1.3) *satisfying* (BC2) *and* (BC3), *respectively.*

(3) *If* $(\gamma, \delta) \in \Omega_1$ *and* $\gamma = 0$, $\delta \neq 0$, *then there are infinitely many monotonic nondecreasing and nonincreasing standing wave solutions of* (1.3) *satisfying* (BC1) *and* (BC4), *respectively*.
(4) *If* $(\gamma, \delta) \in \Omega_2$ *and* $\gamma \neq 0$, $\delta = 0$, *then there are infinitely many monotonic nonincreasing and nondecreasing standing wave solutions of* (1.3) *satisfying* (BC2) *and* (BC3), *respectively*.

Finally, let us remark that in this paper we are concerned with the existence of monotonic traveling wave and standing wave solutions. The issue of the stability of these solutions is still open and will be discussed in a future paper.

The remainder of this paper is organized as follows. In Section 2, with the aid of real roots of the corresponding characteristic function of (1.5) at x_1^+, we construct the upper and lower solutions of (1.5) for all $(\gamma, \delta) \in \Omega_i$, $i = 1, 2, 3, 4$. We then prove Theorem 1 by the method of monotone iteration with the constructed upper and lower solutions. In Section 3, we first establish a discrete version of the monotone iteration scheme and then prove the existence results as stated in parts (1) and (2) of Theorem 2. When the profile equation (1.6) is the purely delayed or advanced type, we prove parts (3) and (4) of Theorem 2 by using the theory of dynamical systems for maps. Finally, in Section 4, some numerical results of the monotone iteration scheme for traveling wave solutions are presented.

2. Monotonic traveling wave solutions $(c \neq 0)$

In this section, we will first introduce and study the characteristic function of (1.5) at x_1^+. Making use of the real roots of the characteristic function, we can construct the upper and lower solutions of (1.5) for all $(\gamma, \delta) \in \Omega_i$, $i = 1, 2, 3, 4$. We then prove Theorem 1 by the monotone iteration method.

2.1. *Properties of the characteristic function*

First of all, we define the characteristic function of (1.5) at x_1^+ by

$$\triangle(\sigma, c; x_1^+) = -c\sigma + \alpha + 2\beta - a - \gamma e^{-\sigma} - \delta e^{\sigma}. \tag{2.1}$$

The characteristic function (2.1) arises from the linearized equation of (1.5) at the equilibrium solution x_1^+ and its roots play crucial roles in studying the behavior of solutions of (1.5) near x_1^+. Some properties of the characteristic function are stated in the following lemma.

Lemma 2.1. *Assume that* (1.7) *holds. Then we have*

(1) *If* $(\gamma, \delta) \in \Omega_1$, *then there exist* $c_3 < c_1 < c_5 = 0$, $\varepsilon_1 > 0$, $\varepsilon_3 > 0$, $\varepsilon_5 > 0$, $\sigma_1 := \sigma_1(c) < 0$, $\sigma_3 := \sigma_3(c) > 0$, *and* $\sigma_5 := \sigma_5(c) < 0$, *such that*

$$\triangle(\sigma_1, c; x_1^+) = 0, \;\; \triangle(\sigma_1 - \varepsilon, c; x_1^+) > 0, \;\text{ if }\; c_1 < c < 0, \;\; 0 < \varepsilon < \varepsilon_1,$$

$$\triangle(\sigma_3, c; x_1^+) = 0, \;\; \triangle(\sigma_3 + \varepsilon, c; x_1^+) > 0, \;\text{ if }\; c < c_3 < 0, \;\; 0 < \varepsilon < \varepsilon_3,$$

$$\triangle(\sigma_5, c; x_1^+) = 0, \;\; \triangle(\sigma_5 - \varepsilon, c; x_1^+) > 0, \;\text{ if }\; 0 = c_5 < c, \;\; 0 < \varepsilon < \varepsilon_5.$$

(2) *If* $(\gamma, \delta) \in \Omega_2$, *then there exist* $c_9 > c_7 > c_{11} = 0$, $\varepsilon_7 > 0$, $\varepsilon_9 > 0$, $\varepsilon_{11} > 0$, $\sigma_7 := \sigma_7(c) > 0$, $\sigma_9 := \sigma_9(c) < 0$, *and* $\sigma_{11} := \sigma_{11}(c) > 0$, *such that*

$$\triangle(\sigma_7, c; x_1^+) = 0, \;\; \triangle(\sigma_7 + \varepsilon, c; x_1^+) > 0, \;\text{ if }\; 0 < c < c_7, \;\; 0 < \varepsilon < \varepsilon_7,$$

$$\triangle(\sigma_9, c; x_1^+) = 0, \;\; \triangle(\sigma_9 - \varepsilon, c; x_1^+) > 0, \;\text{ if }\; 0 < c_9 < c, \;\; 0 < \varepsilon < \varepsilon_9,$$

$$\triangle(\sigma_{11}, c; x_1^+) = 0, \;\; \triangle(\sigma_{11} + \varepsilon, c; x_1^+) > 0, \;\text{ if }\; c < c_{11} = 0, \;\; 0 < \varepsilon < \varepsilon_{11}.$$

(3) *If* $(\gamma, \delta) \in \Omega_3$, *then there exist* $c_{15} < 0 < c_{13}$, $\varepsilon_{13} > 0$, $\varepsilon_{15} > 0$, $\sigma_{13} := \sigma_{13}(c) < 0$ *and* $\sigma_{15} := \sigma_{15}(c) > 0$ *such that*

$$\triangle(\sigma_{13}, c; x_1^+) = 0, \;\; \triangle(\sigma_{13} - \varepsilon, c; x_1^+) > 0, \;\text{ if }\; 0 < c_{13} < c, \;\; 0 < \varepsilon < \varepsilon_{13},$$

$$\triangle(\sigma_{15}, c; x_1^+) = 0, \;\; \triangle(\sigma_{15} + \varepsilon, c; x_1^+) > 0, \;\text{ if }\; c < c_{15} < 0, \;\; 0 < \varepsilon < \varepsilon_{15}.$$

(4) *If* $(\gamma, \delta) \in \Omega_4$, *then there exist* $c_{17} < 0 < c_{19}$, $\varepsilon_{17} > 0$, $\varepsilon_{19} > 0$, $\sigma_{17} := \sigma_{17}(c) > 0$ *and* $\sigma_{19} := \sigma_{19}(c) < 0$ *such that*

$$\triangle(\sigma_{17}, c; x_1^+) = 0, \;\; \triangle(\sigma_{17} + \varepsilon, c; x_1^+) > 0, \;\text{ if }\; c < c_{17} < 0, \;\; 0 < \varepsilon < \varepsilon_{17},$$

$$\triangle(\sigma_{19}, c; x_1^+) = 0, \;\; \triangle(\sigma_{19} - \varepsilon, c; x_1^+) > 0, \;\text{ if }\; 0 < c_{19} < c, \;\; 0 < \varepsilon < \varepsilon_{19}.$$

Proof. We first define an auxiliary function $h: \mathbf{R} \to \mathbf{R}$ by

$$h(\sigma) = -(\alpha + 2\beta) + a + \gamma e^{-\sigma} + \delta e^{\sigma}.$$

Then we have

$$h(0) = -(\alpha + 2\beta) + a + \gamma + \delta > 0, \tag{2.2}$$

$$h'(0) = \delta - \gamma, \tag{2.3}$$

$$h''(\sigma) = \gamma e^{-\sigma} + \delta e^{\sigma}, \tag{2.4}$$

and

$$\lim_{\sigma \to -\infty} h(\sigma) = \begin{cases} \infty & \text{if } \gamma > 0, \\ -(\alpha + 2\beta) + a & \text{if } \gamma = 0, \end{cases} \tag{2.5}$$

$$\lim_{\sigma \to \infty} h(\sigma) = \begin{cases} \infty & \text{if } \delta > 0, \\ -(\alpha + 2\beta) + a & \text{if } \delta = 0. \end{cases} \tag{2.6}$$

Therefore, if γ and δ are not both zero, then $h(\sigma)$ is a convex function. Notice that finding a root of $\triangle(\sigma, c; x_1^+) = 0$ is equivalent to finding a root of $-c\sigma = h(\sigma)$. With this in mind, we now divide the proof into the following four parts:

(1) If $(\gamma, \delta) \in \Omega_1$ and $\gamma > 0$, then $h'(0) > 0$ and $h(\sigma)$ has an unique minimum value which occurs at some $\sigma^* < 0$ such that $h(\sigma^*) = -(\alpha + 2\beta) + 2\sqrt{\gamma\delta} + a < 0$. Hence, there exist $c_3 < c_1 < 0$ such that both the lines $y = c_1\sigma$ and $y = c_3\sigma$ are tangent to $y = h(\sigma)$. Now, the equation $-c\sigma = h(\sigma)$ has two real roots whenever $c_1 < c < 0$, $0 = c_5 < c$, or $c < c_3 < 0$. Denote the larger one of two negative roots of $-c\sigma = h(\sigma)$ by σ_1 and σ_5 for $c_1 < c < 0$ and $0 = c_5 < c$, respectively, and denote the smaller positive root of $-c\sigma = h(\sigma)$ by σ_3 if $c < c_3 < 0$. Then, evidently, there exist $\varepsilon_1 > 0$, $\varepsilon_3 > 0$, and $\varepsilon_5 > 0$ such that the assertions hold. If $\gamma = 0$, then $h(\sigma)$ is increasing in σ and $\lim_{\sigma \to -\infty} h(\sigma) = a - (\alpha + 2\beta) < 0$. We still can achieve the proof with a minor modification.

(2) The argument for the case $(\gamma, \delta) \in \Omega_2$ is similar to part (1).

(3) If $(\gamma, \delta) \in \Omega_3$ and $\gamma > 0$, then $h'(0) > 0$ and $h(\sigma)$ has an unique minimum value which occurs at some $\sigma^* < 0$ such that $h(\sigma^*) = -(\alpha + 2\beta) + 2\sqrt{\gamma\delta} + a > 0$. Thus, there exist $c_{15} < 0 < c_{13}$ such that both the lines $y = c_{13}\sigma$ and $y = c_{15}\sigma$ are tangent to $y = h(\sigma)$. The equation $-c\sigma = h(\sigma)$ has two real roots whenever $0 < c_{13} < c$ or $c < c_{15} < 0$. Let $\sigma_{13} < 0$ and $\sigma_{15} > 0$ be the larger one and smaller one of the two roots in each case. Then there exist $\varepsilon_{13} > 0$ and $\varepsilon_{15} > 0$ such that the assertions hold. Finally, for the case $a \geqslant \alpha + 2\beta$, γ is allowed to be zero. If $\gamma = 0$, then $h(\sigma)$ is increasing in σ and $\lim_{\sigma \to -\infty} h(\sigma) = a - (\alpha + 2\beta) \geqslant 0$. One can conclude the assertions still hold.

(4) For $(\gamma, \delta) \in \Omega_4$, the proof can be achieved in a similar way.

This completes the proof. $\square$

2.2. Construction of upper and lower solutions

In this subsection, we will construct upper and lower solutions of (1.5) to establish the existence of traveling wave solutions of system (1.3).

Definition 2.2. A continuous function $U: \mathbf{R} \to \mathbf{R}$ is called an upper solution of (1.5) if it is differentiable almost everywhere and satisfies

$$-cU'(s) \geqslant -g(U(s)) + aU(s) + \gamma U(s-1) + \delta U(s+1) \quad \text{a.e.} \tag{2.7}$$

A lower solution $L: \mathbf{R} \to \mathbf{R}$ of (1.5) is defined in a similar way by reversing the inequality, that is,

$$-cL'(s) \leqslant -g(L(s)) + aL(s) + \gamma L(s-1) + \delta L(s+1) \quad \text{a.e.} \tag{2.8}$$

By the definition of upper and lower solutions, we immediately have the following properties.

Lemma 2.3. *Assume that* $U(s)$ *and* $L(s)$ *are upper and lower solutions of* (1.5), *respectively. Define*

$$(\widehat{U}(s), \widehat{c}, \widehat{a}, \widehat{\gamma}, \widehat{\delta}) := (U(-s), -c, a, \delta, \gamma), \tag{2.9}$$

$$(\widehat{L}(s), \widehat{c}, \widehat{a}, \widehat{\gamma}, \widehat{\delta}) := (L(-s), -c, a, \delta, \gamma). \tag{2.10}$$

Then $\widehat{U}(s)$ *and* $\widehat{L}(s)$, *respectively, satisfy the following inequalities*:

$$-\widehat{c}\widehat{U}'(s) \geqslant -g(\widehat{U}(s)) + \widehat{a}\widehat{U}(s) + \widehat{\gamma}\widehat{U}(s-1) + \widehat{\delta}\widehat{U}(s+1) \quad \text{a.e.} \tag{2.11}$$

$$-\widehat{c}\widehat{L}'(s) \leqslant -g(\widehat{L}(s)) + \widehat{a}\widehat{L}(s) + \widehat{\gamma}\widehat{L}(s-1) + \widehat{\delta}\widehat{L}(s+1) \quad \text{a.e.} \tag{2.12}$$

That is, $\widehat{U}(s)$ *and* $\widehat{L}(s)$ *are, respectively, upper and lower solutions to the profile equation*

$$-\widehat{c}\varphi'(s) = -g(\varphi(s)) + \widehat{a}\varphi(s) + \widehat{\gamma}\varphi(s-1) + \widehat{\delta}\varphi(s+1).$$

Proof. The proof is a direct verification. □

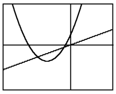
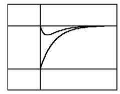
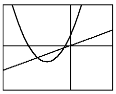
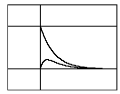
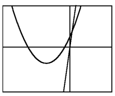
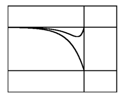
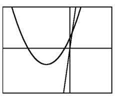
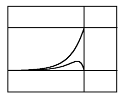
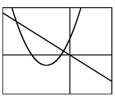
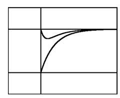
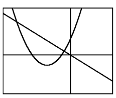
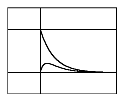
Now, according to the properties of the characteristic function stated in Lemma 2.1, we can construct the upper and lower solutions of (1.5) as following:

Proposition 2.4. *Assume that* (1.7) *holds.*

(1) *Let* $(\gamma, \delta) \in \Omega_1$ *and* $\zeta > 0$ *be small enough. Then each component of the pair of functions* $(U_i(s), L_i(s))$, *for* $i = 1, 2, \ldots, 6$, *as shown in* Table 5, *are upper and lower solutions of* (1.5), *respectively, where* $c_1 < c < 0$ *and* $0 < \varepsilon < \varepsilon_1$ *for* $i = 1, 2$, $c < c_3 < 0$ *and* $0 < \varepsilon < \varepsilon_3$ *for* $i = 3, 4$, *and* $0 = c_5 < c$ *and* $0 < \varepsilon < \varepsilon_5$ *for* $i = 5, 6$.
(2) *Let* $(\gamma, \delta) \in \Omega_2$ *and* $\zeta > 0$ *be small enough. Then each component of the pair of functions* $(U_i(s), L_i(s))$, *for* $i = 7, 8, \ldots, 12$, *as shown in* Table 6, *are upper and lower solutions of* (1.5), *respectively, where* $0 < c < c_7$ *and* $0 < \varepsilon < \varepsilon_7$ *for* $i = 7, 8$, $0 < c_9 < c$ *and* $0 < \varepsilon < \varepsilon_9$ *for* $i = 9, 10$, *and* $c < c_{11} = 0$ *and* $0 < \varepsilon < \varepsilon_{11}$ *for* $i = 11, 12$.
(3) *Let* $(\gamma, \delta) \in \Omega_3$ *and* $\zeta > 0$ *be small enough. Then each component of the pair of functions* $(U_i(s), L_i(s))$, *for* $i = 13, \ldots, 16$, *as shown in* Table 7, *are upper and lower solutions of* (1.5), *respectively, where* $0 < c_{13} < c$ *and* $0 < \varepsilon < \varepsilon_{13}$ *for* $i = 13, 14$, *and* $c < c_{15} < 0$ *and* $0 < \varepsilon < \varepsilon_{15}$ *for* $i = 15, 16$.
(4) *Let* $(\gamma, \delta) \in \Omega_4$ *and* $\zeta > 0$ *be small enough. Then each component of the pair of functions* $(U_i(s), L_i(s))$, *for* $i = 17, \ldots, 20$, *as shown in* Table 8, *are upper and lower solutions of* (1.5), *respectively, where* $c < c_{17} < 0$ *and* $0 < \varepsilon < \varepsilon_{17}$ *for* $i = 17, 18$, *and* $0 < c_{19} < c$ *and* $0 < \varepsilon < \varepsilon_{19}$ *for* $i = 19, 20$.

Proof. To prove the assertions, we only need to show that $(U_1(s), L_1(s))$ and $(U_2(s), L_2(s))$ are upper–lower solution pairs of (1.5). For other cases, by combining

Table 5
Formulas of upper–lower solution pairs $(U_i(s), L_i(s))$ for $i = 1$ to 6

Shapes of $y = h(\sigma), -c\sigma$	Explicit forms of $U_i(s)$ and $L_i(s)$	Shapes of $U_i(s), L_i(s)$
	$U_1(s) = \begin{cases} x_1^+ - \zeta(1-e^{-\varepsilon s})e^{\sigma_1 s}, & s \geqslant 0, \\ x_1^+, & s \leqslant 0. \end{cases}$ $L_1(s) = \begin{cases} x_1^+ - x_1^+ e^{\sigma_1 s}, & s \geqslant 0, \\ 0, & s \leqslant 0. \end{cases}$	
	$U_2(s) = \begin{cases} x_1^+ + (x_2^+ - x_1^+)e^{\sigma_1 s}, & s \geqslant 0, \\ x_2^+, & s \leqslant 0. \end{cases}$ $L_2(s) = \begin{cases} x_1^+ + \zeta(1-e^{-\varepsilon s})e^{\sigma_1 s}, & s \geqslant 0, \\ x_1^+. & s \leqslant 0. \end{cases}$	
	$U_3(s) = \begin{cases} x_1^+, & s \geqslant 0, \\ x_1^+ - \zeta(1-e^{\varepsilon s})e^{\sigma_3 s}, & s \leqslant 0. \end{cases}$ $L_3(s) = \begin{cases} 0, & s \geqslant 0, \\ x_1^+ - x_1^+ e^{\sigma_3 s}, & s \leqslant 0. \end{cases}$	
	$U_4(s) = \begin{cases} x_2^+, & s \geqslant 0, \\ x_1^+ + (x_2^+ - x_1^+)e^{\sigma_3 s}, & s \leqslant 0. \end{cases}$ $L_4(s) = \begin{cases} x_1^+, & s \geqslant 0, \\ x_1^+ + \zeta(1-e^{\varepsilon s})e^{\sigma_3 s}, & s \leqslant 0. \end{cases}$	
	$U_5(s) = \begin{cases} x_1^+ - \zeta(1-e^{-\varepsilon s})e^{\sigma_5 s}, & s \geqslant 0, \\ x_1^+, & s \leqslant 0. \end{cases}$ $L_5(s) = \begin{cases} x_1^+ - x_1^+ e^{\sigma_5 s}, & s \geqslant 0, \\ 0, & s \leqslant 0. \end{cases}$	
	$U_6(s) = \begin{cases} x_1^+ + (x_2^+ - x_1^+)e^{\sigma_5 s}, & s \geqslant 0, \\ x_2^+, & s \leqslant 0. \end{cases}$ $L_6(s) = \begin{cases} x_1^+ + \zeta(1-e^{-\varepsilon s})e^{\sigma_5 s}, & s \geqslant 0, \\ x_1^+, & s \leqslant 0. \end{cases}$	

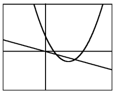
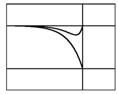
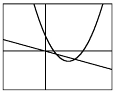
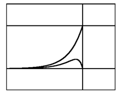
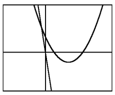
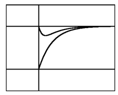
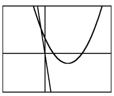
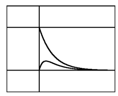
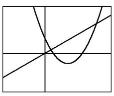
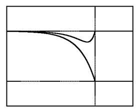
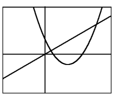
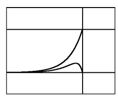
Lemma 2.3 with similar arguments, we can prove the results according to the procedure as shown in Fig. 4.

In Fig. 4, the notation $i \leftrightarrow j$ means that $(U_j(s), L_j(s))$ can be proved as an upper–lower solution pair by the same method as in proving the upper–lower solution pair $(U_i(s), L_i(s))$, and vice versa. While the notation $i \to j$ means that $(U_j(s), L_j(s))$ can be proved to be an upper–lower solution pair by Lemma 2.3 when $(U_i(s), L_i(s))$ is already known an upper–lower solution pair.

(i) $(U_1(s), L_1(s))$ *is an upper–lower solution pair of* (1.5). By the definition of $U_1(s)$ as stated in Table 5, if $s<0$ then $-cU_1'(s) = 0$ and, since $g(x_1^+) = (a+\gamma+\delta)x_1^+$, we have

$$\begin{aligned} & -g(U_1(s)) + aU_1(s) + \gamma U_1(s-1) + \delta U_1(s+1) \\ &= -g(x_1^+) + ax_1^+ + \gamma x_1^+ + \delta U_1(s+1) \end{aligned}$$

Table 6
Formulas of upper–lower solution pairs $(U_i(s), L_i(s))$ for $i = 7$ to 12

Shapes of $y = h(\sigma), -c\sigma$	Explicit forms of $U_i(s)$ and $L_i(s)$	Shapes of $U_i(s), L_i(s)$
	$U_7(s) = \begin{cases} x_1^+, & s \geqslant 0, \\ x_1^+ - \zeta(1 - e^{\varepsilon s})e^{\sigma_7 s}, & s \leqslant 0. \end{cases}$ $L_7(s) = \begin{cases} 0, & s \geqslant 0, \\ x_1^+ - x_1^+ e^{\sigma_7 s}, & s \leqslant 0. \end{cases}$	
	$U_8(s) = \begin{cases} x_2^+, & s \geqslant 0, \\ x_1^+ + (x_2^+ - x_1^+)e^{\sigma_7 s}, & s \leqslant 0. \end{cases}$ $L_8(s) = \begin{cases} x_1^+, & s \geqslant 0, \\ x_1^+ + \zeta(1 - e^{\varepsilon s})e^{\sigma_7 s}, & s \leqslant 0. \end{cases}$	
	$U_9(s) = \begin{cases} x_1^+ - \zeta(1 - e^{-\varepsilon s})e^{\sigma_9 s}, & s \geqslant 0, \\ x_1^+, & s \leqslant 0. \end{cases}$ $L_9(s) = \begin{cases} x_1^+ - x_1^+ e^{\sigma_9 s}, & s \geqslant 0, \\ 0, & s \leqslant 0. \end{cases}$	
	$U_{10}(s) = \begin{cases} x_1^+ + (x_2^+ - x_1^+)e^{\sigma_9 s}, & s \geqslant 0, \\ x_2^+, & s \leqslant 0. \end{cases}$ $L_{10}(s) = \begin{cases} x_1^+ + \zeta(1 - e^{-\varepsilon s})e^{\sigma_9 s}, & s \geqslant 0, \\ x_1^+, & s \leqslant 0. \end{cases}$	
	$U_{11}(s) = \begin{cases} x_1^+, & s \geqslant 0, \\ x_1^+ - \zeta(1 - e^{\varepsilon s})e^{\sigma_{11} s}, & s \leqslant 0. \end{cases}$ $L_{11}(s) = \begin{cases} 0, & s \geqslant 0, \\ x_1^+ - x_1^+ e^{\sigma_{11} s}, & s \leqslant 0. \end{cases}$	
	$U_{12}(s) = \begin{cases} x_2^+, & s \geqslant 0, \\ x_1^+ + (x_2^+ - x_1^+)e^{\sigma_{11} s}, & s \leqslant 0. \end{cases}$ $L_{12}(s) = \begin{cases} x_1^+, & s \geqslant 0, \\ x_1^+ + \zeta(1 - e^{\varepsilon s})e^{\sigma_{11} s}, & s \leqslant 0. \end{cases}$	

$$\begin{aligned} &\leqslant -g(x_1^+) + ax_1^+ + \gamma x_1^+ + \delta x_1^+ \\ &= -cU_1'(s). \end{aligned} \tag{2.13}$$

If $s > 0$, then using the fact that $\triangle(\sigma_1, c; x_1^+) = 0$ we obtain

$$\begin{aligned} -cU_1'(s) &= \zeta c \sigma_1 e^{\sigma_1 s} - \zeta c(\sigma_1 - \varepsilon)e^{(\sigma_1 - \varepsilon)s} \\ &= \zeta(\alpha + 2\beta - a - \gamma e^{-\sigma_1} - \delta e^{\sigma_1})e^{\sigma_1 s} - \zeta c(\sigma_1 - \varepsilon)e^{(\sigma_1 - \varepsilon)s} \end{aligned} \tag{2.14}$$

and

$$-g(U_1(s)) + aU_1(s) + \gamma U_1(s-1) + \delta U_1(s+1)$$

Table 7
Formulas of upper–lower solution pairs $(U_i(s), L_i(s))$ for $i = 13$ to 16

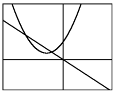
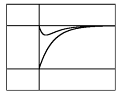
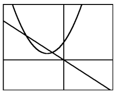
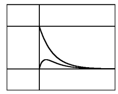
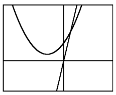
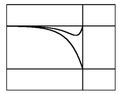
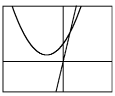
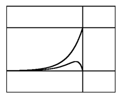
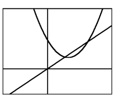
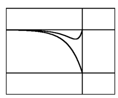
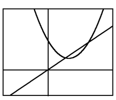
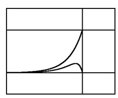
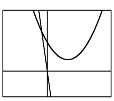
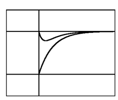
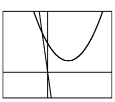
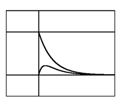
Shapes of $y = h(\sigma), -c\sigma$	Explicit forms of $U_i(s)$ and $L_i(s)$	Shapes of $U_i(s), L_i(s)$
	$U_{13}(s) = \begin{cases} x_1^+ - \zeta(1 - e^{-\varepsilon s})e^{\sigma_{13} s}, & s \geqslant 0, \\ x_1^+, & s \leqslant 0. \end{cases}$ $L_{13}(s) = \begin{cases} x_1^+ - x_1^+ e^{\sigma_{13} s}, & s \geqslant 0, \\ 0, & s \leqslant 0. \end{cases}$	
	$U_{14}(s) = \begin{cases} x_1^+ + (x_2^+ - x_1^+)e^{\sigma_{13} s}, & s \geqslant 0, \\ x_2^+, & s \leqslant 0. \end{cases}$ $L_{14}(s) = \begin{cases} x_1^+ + \zeta(1 - e^{-\varepsilon s})e^{\sigma_{13} s}, & s \geqslant 0, \\ x_1^+, & s \leqslant 0. \end{cases}$	
	$U_{15}(s) = \begin{cases} x_1^+, & s \geqslant 0, \\ x_1^+ - \zeta(1 - e^{\varepsilon s})e^{\sigma_{15} s}, & s \leqslant 0. \end{cases}$ $L_{15}(s) = \begin{cases} 0, & s \geqslant 0, \\ x_1^+ - x_1^+ e^{\sigma_{15} s}, & s \leqslant 0. \end{cases}$	
	$U_{16}(s) = \begin{cases} x_2^+, & s \geqslant 0, \\ x_1^+ + (x_2^+ - x_1^+)e^{\sigma_{15} s}, & s \leqslant 0. \end{cases}$ $L_{16}(s) = \begin{cases} x_1^+, & s \geqslant 0, \\ x_1^+ + \zeta(1 - e^{\varepsilon s})e^{\sigma_{15} s}, & s \leqslant 0. \end{cases}$	

Table 8
Formulas of upper–lower solution pairs $(U_i(s), L_i(s))$ for $i = 17$ to 20

Shapes of $y = h(\sigma), -c\sigma$	Explicit forms of $U_i(s)$ and $L_i(s)$	Shapes of $U_i(s), L_i(s)$
	$U_{17}(s) = \begin{cases} x_1^+, & s \geqslant 0, \\ x_1^+ - \zeta(1 - e^{\varepsilon s})e^{\sigma_{17} s}, & s \leqslant 0. \end{cases}$ $L_{17}(s) = \begin{cases} 0, & s \geqslant 0, \\ x_1^+ - x_1^+ e^{\sigma_{17} s}, & s \leqslant 0. \end{cases}$	
	$U_{18}(s) = \begin{cases} x_2^+, & s \geqslant 0, \\ x_1^+ + (x_2^+ - x_1^+)e^{\sigma_{17} s}, & s \leqslant 0. \end{cases}$ $L_{18}(s) = \begin{cases} x_1^+, & s \geqslant 0, \\ x_1^+ + \zeta(1 - e^{\varepsilon s})e^{\sigma_{17} s}, & s \leqslant 0. \end{cases}$	
	$U_{19}(s) = \begin{cases} x_1^+ - \zeta(1 - e^{-\varepsilon s})e^{\sigma_{19} s}, & s \geqslant 0, \\ x_1^+, & s \leqslant 0. \end{cases}$ $L_{19}(s) = \begin{cases} x_1^+ - x_1^+ e^{\sigma_{19} s}, & s \geqslant 0, \\ 0, & s \leqslant 0. \end{cases}$	
	$U_{20}(s) = \begin{cases} x_1^+ + (x_2^+ - x_1^+)e^{\sigma_{19} s}, & s \geqslant 0, \\ x_2^+, & s \leqslant 0. \end{cases}$ $L_{20}(s) = \begin{cases} x_1^+ + \zeta(1 - e^{-\varepsilon s})e^{\sigma_{19} s}, & s \geqslant 0, \\ x_1^+, & s \leqslant 0. \end{cases}$	

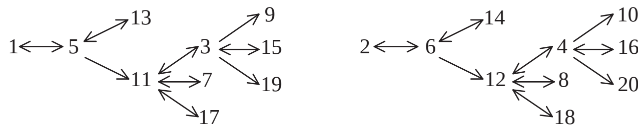


Fig. 4. Procedure for proving the upper–lower solution pairs.

$$
\begin{aligned}
&= -g(x_1^+ - \zeta(1-e^{-\varepsilon s})e^{\sigma_1 s}) + a\{x_1^+ - \zeta(1-e^{-\varepsilon s})e^{\sigma_1 s}\} \\
&\quad + \gamma U_1(s-1) + \delta\{x_1^+ - \zeta(1-e^{-\varepsilon(s+1)})e^{\sigma_1(s+1)}\} \\
&= g(x_1^+) - g(x_1^+ - \zeta(1-e^{-\varepsilon s})e^{\sigma_1 s}) - a\zeta(1-e^{-\varepsilon s})e^{\sigma_1 s} \\
&\quad + \gamma U_1(s-1) - \delta\zeta(1-e^{-\varepsilon(s+1)})e^{\sigma_1(s+1)} - \gamma x_1^+. \qquad (2.15)
\end{aligned}
$$

According to the characteristic function (2.1), we know that

$$\triangle(\sigma_1 - \varepsilon, c; x_1^+) = -c(\sigma_1 - \varepsilon) + \alpha + 2\beta - a - \gamma e^{-(\sigma_1-\varepsilon)} - \delta e^{\sigma_1 - \varepsilon}. \qquad (2.16)$$

Thus, combining (2.14)–(2.16) with sufficiently small $\zeta > 0$, one can verify that $U_1(s)$ is an upper solution of (1.5) for $s > 0$ if and only if

$$
\begin{aligned}
\gamma U_1(s-1) \leqslant{} & \zeta \triangle(\sigma_1 - \varepsilon, c; x_1^+)e^{(\sigma_1-\varepsilon)s} \\
& + \gamma\{x_1^+ - \zeta(1-e^{-\varepsilon(s-1)})e^{\sigma_1(s-1)}\}. \qquad (2.17)
\end{aligned}
$$

Since $\triangle(\sigma_1 - \varepsilon, c; x_1^+) > 0$ and for $s \geqslant 1$,

$$U_1(s-1) = x_1^+ - \zeta(1-e^{-\varepsilon(s-1)})e^{\sigma_1(s-1)},$$

and then (2.17) holds obviously. Otherwise, if $0 < s \leqslant 1$ then

$$
\begin{aligned}
&\gamma\{x_1^+ - \zeta(1-e^{-\varepsilon(s-1)})e^{\sigma_1(s-1)}\} - \gamma U_1(s-1) \\
&\quad = \gamma\zeta(e^{-\varepsilon(s-1)} - 1)e^{\sigma_1(s-1)} \\
&\quad \geqslant 0. \qquad (2.18)
\end{aligned}
$$

Therefore (2.17) follows and thus $U_1(s)$ is an upper solution of (1.5).

For $L_1(s)$, if $s < 0$ then $-cL_1{}'(s) = 0$ and we have

$$
\begin{aligned}
&-g(L_1(s)) + aL_1(s) + \gamma L_1(s-1) + \delta L_1(s+1) \\
&\quad = \delta L_1(s+1) \\
&\quad \geqslant -cL_1{}'(s). \qquad (2.19)
\end{aligned}
$$

On the other hand, if $s>0$ then using that facts $\triangle(\sigma_1, c; x_1^+) = 0$ and $g(x_1^+) = (a+\gamma+\delta)x_1^+$ again, we get

$$\begin{aligned} -cL_1{}'(s) &= c\sigma_1 x_1^+ e^{\sigma_1 s} \\ &= (\alpha+2\beta)x_1^+ e^{\sigma_1 s} - x_1^+(a+\delta e^{\sigma_1})e^{\sigma_1 s} - \gamma x_1^+ e^{\sigma_1 (s-1)} \end{aligned} \tag{2.20}$$

and

$$\begin{aligned} &-g(L_1(s)) + aL_1(s) + \gamma L_1(s-1) + \delta L_1(s+1) \\ &\quad = -g(x_1^+ - x_1^+ e^{\sigma_1 s}) + a(x_1^+ - x_1^+ e^{\sigma_1 s}) + \gamma L_1(s-1) + \delta(x_1^+ - x_1^+ e^{\sigma_1 (s+1)}) \\ &\quad = g(x_1^+) - g(x_1^+ - x_1^+ e^{\sigma_1 s}) - \gamma x_1^+ - x_1^+(a+\delta e^{\sigma_1})e^{\sigma_1 s} + \gamma L_1(s-1). \end{aligned} \tag{2.21}$$

Now, combining (2.20) with (2.21), we can conclude that $L_1(s)$ is a lower solution of (1.5) for $s>0$ if and only if

$$\begin{aligned} g(x_1^+) - g(x_1^+ - x_1^+ e^{\sigma_1 s}) \geqslant\ & (\alpha+2\beta)x_1^+ e^{\sigma_1 s} \\ & + \gamma(x_1^+ - x_1^+ e^{\sigma_1 (s-1)} - L_1(s-1)). \end{aligned} \tag{2.22}$$

Since $\gamma(x_1^+ - x_1^+ e^{\sigma_1 (s-1)} - L_1(s-1)) \leqslant 0$ for all $s>0$, for proving $L_1(s)$ is a lower solution of (1.5) for $s>0$, it suffices to prove that

$$g(x_1^+) - g(x_1^+ - x_1^+ e^{\sigma_1 s}) \geqslant (\alpha+2\beta)x_1^+ e^{\sigma_1 s}. \tag{2.23}$$

If $V_p < x_1^+ - x_1^+ e^{\sigma_1 s} < V_v$ then, by (1.2), we have

$$g(x_1^+) - g(x_1^+ - x_1^+ e^{\sigma_1 s}) = (\alpha+2\beta)x_1^+ e^{\sigma_1 s}. \tag{2.24}$$

Otherwise, if $0 < x_1^+ - x_1^+ e^{\sigma_1 s} \leqslant V_p$ then, since $\beta<0$, we obtain

$$\begin{aligned} g(x_1^+) - g(x_1^+ - x_1^+ e^{\sigma_1 s}) &= 2\beta(x_1^+ - V_p) + \alpha x_1^+ e^{\sigma_1 s} \\ &\geqslant (\alpha+2\beta)x_1^+ e^{\sigma_1 s}. \end{aligned} \tag{2.25}$$

Therefore (2.23) follows and $L_1(s)$ is a lower solution of (1.5).

(ii) $(U_2(s), L_2(s))$ *is an upper–lower solution pair of* (1.5).

If $s<0$ then $-cU_2{}'(s) = 0$ and, using the property $g(x_2^+) = (a+\gamma+\delta)x_2^+$, we have

$$\begin{aligned} &-g(U_2(s)) + aU_2(s) + \gamma U_2(s-1) + \delta U_2(s+1) \\ &\quad = -g(x_2^+) + ax_2^+ + \gamma x_2^+ + \delta U_2(s+1) \\ &\quad \leqslant -g(x_2^+) + ax_2^+ + \gamma x_2^+ + \delta x_2^+ \\ &\quad = -cU_2{}'(s). \end{aligned} \tag{2.26}$$

On the other hand, if $s>0$, then using the equality $\triangle(\sigma_1, c; x_1^+) = 0$ we obtain

$$\begin{aligned}
-cU_2'(s) &= -c\sigma_1(x_2^+ - x_1^+)e^{\sigma_1 s} \\
&= -(\alpha + 2\beta)(x_2^+ - x_1^+)e^{\sigma_1 s} + (x_2^+ - x_1^+)(a + \delta e^{\sigma_1})e^{\sigma_1 s} \\
&\quad + \gamma(x_2^+ - x_1^+)e^{\sigma_1(s-1)}, \qquad (2.27)
\end{aligned}$$

and noting that $g(x_1^+) = (a + \gamma + \delta)x_1^+$, we get

$$\begin{aligned}
&-g(U_2(s)) + aU_2(s) + \gamma U_2(s-1) + \delta U_2(s+1) \\
&= -g(x_1^+ + (x_2^+ - x_1^+)e^{\sigma_1 s}) + a(x_1^+ + (x_2^+ - x_1^+)e^{\sigma_1 s}) \\
&\quad + \gamma U_2(s-1) + \delta(x_1^+ + (x_2^+ - x_1^+)e^{\sigma_1(s+1)}) \\
&= g(x_1^+) - g(x_1^+ + (x_2^+ - x_1^+)e^{\sigma_1 s}) - \gamma x_1^+ + \gamma U_2(s-1) \\
&\quad + (x_2^+ - x_1^+)(a + \delta e^{\sigma_1})e^{\sigma_1 s}. \qquad (2.28)
\end{aligned}$$

By (2.27) and (2.28), one can verify that $U_2(s)$ is an upper solution of (1.5) for $s>0$ if and only if

$$\begin{aligned}
&g(x_1^+) - g(x_1^+ + (x_2^+ - x_1^+)e^{\sigma_1 s}) \\
&\quad \leqslant -(\alpha + 2\beta)(x_2^+ - x_1^+)e^{\sigma_1 s} \\
&\qquad + \gamma(x_1^+ + (x_2^+ - x_1^+)e^{\sigma_1(s-1)} - U_2(s-1)). \qquad (2.29)
\end{aligned}$$

Since $\gamma(x_1^+ + (x_2^+ - x_1^+)e^{\sigma_1(s-1)} - U_2(s-1)) \geqslant 0$ for all $s>0$, once we can prove that

$$g(x_1^+) - g(x_1^+ + (x_2^+ - x_1^+)e^{\sigma_1 s}) \leqslant -(\alpha + 2\beta)(x_2^+ - x_1^+)e^{\sigma_1 s}, \qquad (2.30)$$

then inequality (2.29) will hold. If $V_p < x_1^+ + (x_2^+ - x_1^+)e^{\sigma_1 s} \leqslant V_v$ then

$$g(x_1^+) - g(x_1^+ + (x_2^+ - x_1^+)e^{\sigma_1 s}) = -(\alpha + 2\beta)(x_2^+ - x_1^+)e^{\sigma_1 s}. \qquad (2.31)$$

Otherwise, if $V_v < x_1^+ + (x_2^+ - x_1^+)e^{\sigma_1 s}$ then, since $\beta < 0$,

$$\begin{aligned}
g(x_1^+) - g(x_1^+ + (x_2^+ - x_1^+)e^{\sigma_1 s}) &= 2\beta(x_1^+ - V_v) - \alpha(x_2^+ - x_1^+)e^{\sigma_1 s} \\
&\leqslant -(\alpha + 2\beta)(x_2^+ - x_1^+)e^{\sigma_1 s}. \qquad (2.32)
\end{aligned}$$

Therefore (2.30) follows and then $U_2(s)$ is an upper solution of (1.5).

Next, we verify that $L_2(s)$ is a lower solution of (1.5). If $s<0$ then $-cL_2{}'(s)=0$ and

$$\begin{aligned}
&-g(L_2(s))+aL_2(s)+\gamma L_2(s-1)+\delta L_2(s+1)\\
&\quad=-g(x_1^+)+ax_1^+ +\gamma x_1^+ +\delta L_2(s+1)\\
&\quad\geqslant -g(x_1^+)+ax_1^+ +\gamma x_1^+ +\delta x_1^+\\
&\quad=-cL_2{}'(s).
\end{aligned}\tag{2.33}$$

If $s>0$, then using the fact that $-c\sigma_1=-(\alpha+2\beta)+a+\gamma e^{-\sigma_1}+\delta e^{\sigma_1}$, we have

$$\begin{aligned}
-cL_2{}'(s)&=-c\sigma_1\zeta e^{\sigma_1 s}+c(\sigma_1-\varepsilon)\zeta e^{(\sigma_1-\varepsilon)s}\\
&=-\zeta(\alpha+2\beta-a-\gamma e^{-\sigma_1}-\delta e^{\sigma_1})e^{\sigma_1 s}\\
&\quad+c(\sigma_1-\varepsilon)\zeta e^{(\sigma_1-\varepsilon)s}
\end{aligned}\tag{2.34}$$

and, using $g(x_1^+)=(a+\gamma+\delta)x_1^+$, we get

$$\begin{aligned}
&-g(L_2(s))+aL_2(s)+\gamma L_2(s-1)+\delta L_2(s+1)\\
&\quad=-g(x_1^+ +\zeta(1-e^{-\varepsilon s})e^{\sigma_1 s})+a(x_1^+ +\zeta(1-e^{-\varepsilon s})e^{\sigma_1 s})\\
&\qquad+\gamma L_2(s-1)+\delta(x_1^+ +\zeta(1-e^{-\varepsilon(s+1)})e^{\sigma_1(s+1)})\\
&\quad=g(x_1^+)-g(x_1^+ +\zeta(1-e^{-\varepsilon s})e^{\sigma_1 s})-\gamma x_1^+ +a\zeta(1-e^{-\varepsilon s})e^{\sigma_1 s}\\
&\qquad+\gamma L_2(s-1)+\delta\zeta(1-e^{-\varepsilon(s+1)})e^{\sigma_1(s+1)}.
\end{aligned}\tag{2.35}$$

By virtue of the characteristic function (2.1), we know that

$$\triangle(\sigma_1-\varepsilon,c;x_1^+)=-c(\sigma_1-\varepsilon)+\alpha+2\beta-a-\gamma e^{-(\sigma_1-\varepsilon)}-\delta e^{\sigma_1-\varepsilon}.\tag{2.36}$$

Now, combining (2.34)–(2.36), we conclude that $L_2(s)$ is a lower solution of (1.5) for $s>0$ if and only if

$$\zeta\triangle(\sigma_1-\varepsilon,c;x_1^+)e^{(\sigma_1-\varepsilon)s}+\gamma L_2(s-1)\geqslant\gamma(x_1^+ +\zeta(1-e^{-\varepsilon(s-1)})e^{\sigma_1(s-1)}).\tag{2.37}$$

Since $\triangle(\sigma_1-\varepsilon,c;x_1^+)>0$, if $s\geqslant 1$ then

$$\gamma L_2(s-1)=\gamma(x_1^+ +\zeta(1-e^{-\varepsilon(s-1)})e^{\sigma_1(s-1)})$$

which implies that (2.37) holds immediately. Otherwise, if $0<s<1$ then

$$\begin{aligned}&\gamma L_2(s-1) - \gamma(x_1^+ + \zeta(1-e^{-\varepsilon(s-1)})e^{\sigma_1(s-1)})\\&= \gamma\zeta(e^{-\varepsilon(s-1)}-1)e^{\sigma_1(s-1)}\\&\geqslant 0. \qquad (2.38)\end{aligned}$$

Therefore (2.37) follows and hence $L_2(s)$ is a lower solution of (1.5). This completes the proof of Proposition 2.4. □

2.3. Monotone operators

In this subsection, we introduce four monotone operators T_i $(i=1,2,3,4)$ that, combining with the upper and lower solutions constructed in the previous subsection, will be applied to prove Theorem 1 in the next subsection.

Let $C_1 := C(\mathbf{R},[x_0,x_1^+])$ and $C_2 := C(\mathbf{R},[x_1^+,x_2^+])$ be two function spaces of bounded continuous functions defined on $\mathbf{R}$ with different ranges. Then define the mappings H_1 and H_2 on C_1 for $c<0$ and $c>0$, respectively, by

$$H_1(\psi)(s) = -\frac{1}{c}\{-g(\psi(s)) + a\psi(s) + \gamma\psi(s-1) + \delta\psi(s+1)\} + \mu_1\psi(s),$$

$$H_2(\psi)(s) = -\frac{1}{c}\{-g(\psi(s)) + a\psi(s) + \gamma\psi(s-1) + \delta\psi(s+1)\} - \mu_2\psi(s),$$

where $\mu_1,\mu_2>0$ are chosen to be sufficiently large such that $\mu_1>(a-\alpha)/c$ and $\mu_2>(\alpha-a)/c$. Similarly, define the mappings H_3 and H_4 on C_2 for $c<0$ and $c>0$, respectively, by

$$H_3(\psi)(s) = -\frac{1}{c}\{-g(\psi(s)) + a\psi(s) + \gamma\psi(s-1) + \delta\psi(s+1)\} + \mu_3\psi(s),$$

$$H_4(\psi)(s) = -\frac{1}{c}\{-g(\psi(s)) + a\psi(s) + \gamma\psi(s-1) + \delta\psi(s+1)\} - \mu_4\psi(s),$$

where $\mu_3,\mu_4>0$ are also chosen to be sufficiently large such that $\mu_3>(a-\alpha)/c$ and $\mu_4>(\alpha-a)/c$. Now, using the facts that $\gamma\geqslant 0$ and $\delta\geqslant 0$, one can easily verify that the operators H_i $(i=1,2,3,4)$ possess the following monotone properties:

(i) For $c<0$, if $\psi_1(s)\leqslant\psi_2(s)$ then $H_i(\psi_1)(s)\leqslant H_i(\psi_2)(s)$ for $s\in\mathbf{R}$ and $i=1,3$.
(ii) For $c>0$, if $\psi_1(s)\leqslant\psi_2(s)$ then $H_i(\psi_1)(s)\geqslant H_i(\psi_2)(s)$ for $s\in\mathbf{R}$ and $i=2,4$.

Furthermore, it is ready to show that

(i) For $c<0$, $\varphi\in C_j$, $1\leqslant j\leqslant 2$, satisfying (1.5) is equivalent to

$$\varphi(s) = e^{-\mu_i s}\int_{-\infty}^{s} e^{\mu_i t}H_i(\varphi(t))\,dt, \quad s\in\mathbf{R}, \qquad (2.39)$$

where $i=2j-1$;

(ii) For $c>0$, $\varphi \in C_j$, $1 \leqslant j \leqslant 2$, satisfying (1.5) is equivalent to

$$\varphi(s) = -e^{\mu_i s} \int_s^{\infty} e^{-\mu_i t} H_i(\varphi(t))\, dt, \quad s \in \mathbf{R}, \tag{2.40}$$

where $i = 2j$.

We are going to consider (2.39) and (2.40) subject to various asymptotic boundary conditions, (BC1)–(BC4). To this end, we first define the profile spaces $\Gamma_1, \Gamma_2 \subset C_1$ and $\Gamma_3, \Gamma_4 \subset C_2$ by

$$\Gamma_1 = \{\psi \in C_1 \mid \psi \text{ is monotonic nondecreasing and satisfies (BC1)}\},$$

$$\Gamma_2 = \{\psi \in C_1 \mid \psi \text{ is monotonic nonincreasing and satisfies (BC2)}\},$$

$$\Gamma_3 = \{\psi \in C_2 \mid \psi \text{ is monotonic nondecreasing and satisfies (BC3)}\},$$

$$\Gamma_4 = \{\psi \in C_2 \mid \psi \text{ is monotonic nonincreasing and satisfies (BC4)}\},$$

and then define the monotone operators T_1 and T_2 on C_1 for $c<0$ and $c>0$ respectively and, similarly, T_3 and T_4 on C_2, by

$$T_i(\psi)(s) = e^{-\mu_i s} \int_{-\infty}^{s} e^{\mu_i t} H_i(\psi)(t)\, dt, \quad \text{for } i = 1, 3, \tag{2.41}$$

$$T_i(\psi)(s) = -e^{\mu_i s} \int_s^{\infty} e^{-\mu_i t} H_i(\psi)(t)\, dt, \quad \text{for } i = 2, 4. \tag{2.42}$$

Then, by (2.39) and (2.40), it is obvious that $\varphi \in C_j$, $1 \leqslant j \leqslant 2$, is a solution of (1.5) if and only if φ is a fixed point of T_i for $i = 2j-1$ or $2j$ (according as $c<0$ or $c>0$), i.e., $T_i(\varphi)(s) = \varphi(s)$ for all $s \in \mathbf{R}$. We conclude this subsection with some useful properties of the operators T_i that are necessary in the monotone iteration scheme for proving Theorem 1.

Lemma 2.5. *T_i, for $i = 1, 2, 3, 4$, have the following properties*:

(1) *Both T_1 and T_2 are invariant on Γ_i for $i = 1, 2$; both T_3 and T_4 are invariant on Γ_i for $i = 3, 4$. That is, for $j = 1, 2$, $T_j(\psi) \in \Gamma_i$ whenever $\psi \in \Gamma_i$ with $1 \leqslant i \leqslant 2$; for $j = 3, 4$, $T_j(\psi) \in \Gamma_i$ whenever $\psi \in \Gamma_i$ with $3 \leqslant i \leqslant 4$.*
(2) *Assume that $\psi \in C_j$, $1 \leqslant j \leqslant 2$. If $\psi(s)$ is an upper* (*resp., a lower*) *solution of* (1.5), *then $\psi(s) \geqslant$ (resp., $\leqslant$) $T_i(\psi)(s)$ for $i = 2j-1$ or $2j$ according as $c<0$ or $c>0$, and $s \in \mathbf{R}$.*
(3) *If ψ, $\widetilde{\psi}(s) \in C_1$ (resp., C_2) and $\psi(s) \leqslant \widetilde{\psi}(s)$ for $s \in \mathbf{R}$, then $T_i(\psi)(s) \leqslant T_i(\widetilde{\psi})(s)$ for $s \in \mathbf{R}$ and $i = 1, 2$ (resp., $T_i(\psi)(s) \leqslant T_i(\widetilde{\psi})(s)$ for $s \in \mathbf{R}$ and $i = 3, 4$).*
(4) *If $\psi(s) \in C_j$, $1 \leqslant j \leqslant 2$, is an upper* (*resp., a lower*) *solution of* (1.5) *then $T_i(\psi)(s)$, for $i = 2j-1$ or $2j$ according as $c<0$ or $c>0$, is also an upper* (*resp., a lower*) *solution of* (1.5).

Proof. For simplicity, we will focus on the properties of operator T_1 on C_1 with $c<0$. For other cases, the assertions can be proved by using the similar arguments. We omit the details.

(1) By the definition of H_1, we know that $H_1(\psi)(s)$ is monotonic nondecreasing and nonincreasing in variable s for $\psi \in \Gamma_1$ and $\psi \in \Gamma_2$, respectively. Direct calculation shows that

$$T_1(\psi)'(s) = \mu_1 e^{-\mu_1 s} \int_{-\infty}^{s} e^{\mu_1 t}\{H_1(\psi)(s) - H_1(\psi)(t)\}\, dt \tag{2.43}$$

which implies $T_1(\psi)(s)$ is also monotonic nondecreasing and nonincreasing in variable s for $\psi \in \Gamma_1$ and $\psi \in \Gamma_2$, respectively. Since

$$\lim_{s \to -\infty} H_1(\psi)(s) = \mu_1 x_0 \quad \text{and} \quad \lim_{s \to \infty} H_1(\psi)(s) = \mu_1 x_1^+ \quad \text{if } \psi \in \Gamma_1,$$

$$\lim_{s \to -\infty} H_1(\psi)(s) = \mu_1 x_1^+ \quad \text{and} \quad \lim_{s \to \infty} H_1(\psi)(s) = \mu_1 x_0 \quad \text{if } \psi \in \Gamma_2,$$

by l'Hospital's rule, we have

$$\lim_{s \to -\infty} T_1(\psi)(s) = x_0 \quad \text{and} \quad \lim_{s \to \infty} T_1(\psi)(s) = x_1^+ \quad \text{if } \psi \in \Gamma_1,$$

$$\lim_{s \to -\infty} T_1(\psi)(s) = x_1^+ \quad \text{and} \quad \lim_{s \to \infty} T_1(\psi)(s) = x_0 \quad \text{if } \psi \in \Gamma_2.$$

Hence, T_1 is invariant on Γ_1 and Γ_2.

(2) If $\psi \in C_1$ is an upper (resp., a lower) solution of (1.5), by (2.7) (resp., by (2.8)), we have

$$\psi'(s) + \mu_1 \psi(s) \geqslant (\text{resp.,} \ \leqslant) H_1(\psi)(s). \tag{2.44}$$

Integrating (2.44) directly, we obtain

$$\psi(s) \geqslant (\text{resp.,} \ \leqslant) e^{-\mu_1 s} \int_{-\infty}^{s} e^{\mu_1 t} H_1(\psi(t))\, dt = T_1(\psi)(s). \tag{2.45}$$

(3) If ψ, $\widetilde{\psi}(s) \in C_1$ and $\psi(s) \leqslant \widetilde{\psi}(s)$ for $s \in \mathbf{R}$, then $H_1(\psi)(s) \leqslant H_1(\widetilde{\psi})(s)$ for $s \in \mathbf{R}$. Hence, $T_1(\psi)(s) \leqslant T_1(\widetilde{\psi})(s)$ for $s \in \mathbf{R}$.

(4) Assume that $\psi(s) \in C_1$ is an upper (resp., a lower) solution of (1.5). Combining the results of part (2) with the monotonicity of H_1, we have $H_1(\psi)(s) \geqslant$ (resp., $\leqslant$) $H_1(T_1(\psi))(s)$ for $s \in \mathbf{R}$. It turns out that

$$\begin{aligned} -c\frac{dT_1(\psi)}{ds}(s) &= -c\{-\mu_1 T_1(\psi)(s) + H_1(\psi)(s)\} \\ &\geqslant (\text{resp.,} \ \leqslant) -c\{-\mu_1 T_1(\psi)(s) + H_1(T_1(\psi))(s)\} \end{aligned}$$

$$= -g(T_1(\psi)(s)) + aT_1(\psi)(s) + \gamma T_1(\psi)(s-1)$$
$$+ \delta T_1(\psi)(s+1),$$

for $s \in \mathbf{R}$. Hence, $T_1(\psi)(s)$ is an upper (resp., a lower) solution of (1.5).

This completes the proof of Lemma 2.5. $\square$

2.4. *Proof of Theorem 1*

It is now in the position to prove Theorem 1. We first prove case (1-1) in the part (1). From Proposition 2.4, we know that $(U_1(s), L_1(s))$ is an upper–lower solution pair of (1.5). Since L_1 is a monotonic nondecreasing lower solution in Γ_1, by Lemma 2.5, the sequence of monotonic nondecreasing continuous functions $L_1^{(n)}(s) := T_1^{(n)}(L_1)(s)$, for $n = 0, 1, 2, \ldots$, are also lower solutions of (1.5) in Γ_1 and satisfy

$$x_0 \leqslant L_1^{(0)}(s) \leqslant L_1^{(1)}(s) \leqslant L_1^{(2)}(s) \leqslant \cdots \leqslant L_1^{(n)}(s) \leqslant \cdots \leqslant U_1(s) \leqslant x_1^+, \tag{2.46}$$

for all $s \in \mathbf{R}$. Hence, there exists a monotonic nondecreasing function $L_1^* : \mathbf{R} \to \mathbf{R}$ such that

$$L_1^*(s) = \lim_{n \to \infty} L_1^{(n)}(s) \tag{2.47}$$

with $x_0 \leqslant L_1^*(s) \leqslant U_1(s) \leqslant x_1^+$, for all $s \in \mathbf{R}$. Next, for each $s \in \mathbf{R}$, applying Lebesgue's dominated convergence theorem, one can verify that

$$L_1^*(s) = \lim_{n \to \infty} T_1^{(n+1)}(L_1)(s) = \lim_{n \to \infty} T_1(T_1^{(n)}(L_1))(s)$$
$$= \int_{-\infty}^{s} e^{\mu_1(t-s)} H_1(L_1^*)(t)\, dt$$

which implies $L_1^*(s)$ is continuous and then $L_1^*(s) = T_1(L_1^*)(s)$ for all $s \in \mathbf{R}$, i.e., L_1^* is a fixed point of T_1 in C_1. Thus, $L_1^* \in C_1$ is a solution of (1.5). The remaining part is to claim that the limiting function L_1^* satisfies the boundary condition (BC1). To this end, combining the fact that $L_1^*(s)$ is monotonic nondecreasing with the nontrivial barrier function $U_1 \geqslant L_1^*$ on $\mathbf{R}$, one can conclude that L_1^* satisfies (BC1), and this completes the proof of case (1-1).

The proof of other cases can be achieved in a similar way. For example, for case (3-2) in part (3), we iterate the upper solution U_{14} to generate the sequence of upper solutions $U_{14}^{(n)}(s) := T_4^{(n)}(U_{14})(s)$, for $n = 0, 1, 2, \ldots$, and then use $L_{14}(s)$ as the nontrivial barrier function to prove the existence of monotonic nonincreasing solution of (1.5) connecting x_2^+ and x_1^+. $\square$

2.5. *Applications to the discrete Fisher and Nagumo equations*

In this subsection we show that, for some specific templates (a, γ, δ), the RTD-based CNN (1.3) can be reformulated as the discrete Fisher and Nagumo equations.

These typical models possessing traveling wave or standing wave solutions have wide applications in various fields, see, e.g., [1,4,14,19,20].

Consider the one-dimensional RTD-based CNN in the following specific form:

$$\frac{dx_i(t)}{dt} = -g(x_i(t)) + \widehat{a}x_i(t) + \delta(x_{i-1}(t) - 2x_i(t) + x_{i+1}(t)), \tag{2.48}$$

for all $i \in \mathbf{Z}^1$ and $t \in \mathbf{R}$, i.e., take $\gamma = \delta$ and replace the parameter a by $\widehat{a} - 2\delta$ in (1.3). Then the profile equation can be written as

$$-c\varphi'(s) = -g(\varphi(s)) + \widehat{a}\varphi(s) + \delta(\varphi(s-1) - 2\varphi(s) + \varphi(s+1)), \tag{2.49}$$

and condition (1.7) is changed into

$$0 \leqslant \alpha + \frac{2m}{V_v} < \widehat{a} < \alpha \leqslant -2\beta. \tag{2.50}$$

Thus we have five equilibrium solutions to (2.49), namely,

$$x_0 = 0, \quad x_1^{\pm} = \pm\frac{-2\beta V_p}{\widehat{a} - \alpha - 2\beta}, \quad \text{and} \quad x_2^{\pm} = \pm\frac{2\beta(V_v - V_p)}{\widehat{a} - \alpha}. \tag{2.51}$$

Next, shifting and scaling the profile function $\varphi(s)$ and the nonlinearity $-g(x) + \widehat{a}x$ by

$$\psi_1(s) := \frac{\varphi(s) - x_1^+}{x_2^+ - x_1^+}, \tag{2.52}$$

$$f_1(x) := \frac{1}{x_2^+ - x_1^+}\{\widehat{a}(x_1^+ + (x_2^+ - x_1^+)x) - g(x_1^+ + (x_2^+ - x_1^+)x)\}, \tag{2.53}$$

we immediately obtain the following properties:

$$f_1(0) = f_1(1) = 0, \tag{2.54}$$

$$f_1(x) > 0 \text{ for } 0 < x < 1, \quad \text{and} \quad f_1'(0)x \geqslant f(x) \text{ for } x > 0, \tag{2.55}$$

and then it leads to the so-called discrete Fisher equation:

$$-c\psi_1'(s) = f_1(\psi_1(s)) + \delta(\psi_1(s-1) - 2\psi_1(s) + \psi_1(s+1)). \tag{2.56}$$

According to cases (3-2) and (3-4) of Theorem 1, we have actually proved the following results: for given $\widehat{a}$, $\delta \in \mathbf{R}$ with $\delta > 0$,

(i) There exists $c_{14} > 0$ such that if $c > c_{14}$ then we have a monotonic nonincreasing solution $\psi_1(s)$ of (2.56) satisfying the asymptotic boundary conditions, $\lim_{s\to-\infty} \psi_1(s) = 1$ and $\lim_{s\to\infty} \psi_1(s) = 0$;
(ii) There exists $c_{16} < 0$ such that if $c < c_{16}$ then we have a monotonic nondecreasing solution $\psi_1(s)$ of (2.56) satisfying the asymptotic boundary conditions, $\lim_{s\to-\infty} \psi_1(s) = 0$ and $\lim_{s\to\infty} \psi_1(s) = 1$.

Indeed, the discrete Fisher equation (2.56) has been studied by Zinner et. al. [20] in 1993 using different approach (the continuation method) for more general nonlinearity f_1. Here, we briefly summarize their results as follows.

Theorem 2.6 (Zinner et al. [20]). *Suppose f_1 is a Lipschitz continuous function and differentiable at* 0 *satisfying* (2.54) *and* (2.55). *There exists a solution $\psi_1 : \mathbf{R} \to [0,1]$ of* (2.56) *with a given speed $c<0$ satisfying*

$$\lim_{s\to-\infty} \psi_1(s) = 0 \quad \textit{and} \quad \lim_{s\to\infty} \psi_1(s) = 1 \tag{2.57}$$

if and only if

$$\delta \leqslant \sup_{\sigma>0} \frac{-c\sigma - f_1'(0)}{4\sinh^2(\sigma/2)}. \tag{2.58}$$

Furthermore, if (2.58) *holds, then the solution $\psi_1(s)$ is strictly increasing.*

Noting that for our case, $f_1'(0) = \widehat{a} - (\alpha + 2\beta) > 0$. Define the function $h_1(\sigma)$ for $\sigma > 0$ by

$$h_1(\sigma) := \frac{-c\sigma - \widehat{a} + (\alpha+2\beta)}{4\sinh^2(\sigma/2)} = \frac{-c\sigma - \widehat{a} + (\alpha+2\beta)}{e^{\sigma} + e^{-\sigma} - 2}.$$

Then one can verify that $\lim_{\sigma\to0^+} h_1(\sigma) = -\infty$ and $\lim_{\sigma\to\infty} h_1(\sigma) = 0$. Denote

$$\delta^* := \sup_{\sigma>0} \frac{-c\sigma - \widehat{a} + (\alpha+2\beta)}{e^{\sigma} + e^{-\sigma} - 2} < \infty.$$

Then $\delta < \delta^*$ means that, for a fixed $c<0$, there is one positive σ, say λ, such that

$$\delta = h_1(\lambda) = \frac{-c\lambda - \widehat{a} + (\alpha+2\beta)}{e^{\lambda} + e^{-\lambda} - 2}$$

and

$$\delta < h_1(\lambda + \varepsilon) \quad \text{for all sufficiently small } \varepsilon > 0.$$

In other words,

$$-c\lambda - \widehat{a} + (\alpha + 2\beta) - \delta e^{-\lambda} - 2\delta - \delta e^{\lambda} = 0.$$

This exactly means that the characteristic function $\triangle(\sigma, c; x_1^+)$ of (2.49) at x_1^+ has one positive root λ, and it plays the role σ_{15} in Lemma 2.1.

Similarly, if we define

$$\psi_2(s) := \frac{\varphi(s)}{x_2^+}, \tag{2.59}$$

$$f_2(x) := \frac{1}{x_2^+}\{\hat{a}x_2^+ x - g(x_2^+ x)\}, \tag{2.60}$$

then the nonlinear function f_2 possesses the following properties:

$$f_2(0) = f_2\left(\frac{x_1^+}{x_2^+}\right) = f_2(1) = 0, \tag{2.61}$$

$$f_2(x) < 0 \text{ for } 0 < x < \frac{x_1^+}{x_2^+}, \quad \text{and} \quad f_2(x) > 0 \text{ for } \frac{x_1^+}{x_2^+} < x < 1, \tag{2.62}$$

and the profile equation (2.49) is changed into

$$-c{\psi_2}'(s) = f_2(\psi_2(s)) + \delta(\psi_2(s-1) - 2\psi_2(s) + \psi_2(s+1)) \tag{2.63}$$

which is the discrete Nagumo equation. In [19], Zinner proved the following results by using the index-theory:

Theorem 2.7 (Zinner [19]). *Given a wave speed $c<0$, if $V_p + V_v < x_2^+$ then there exists some $\delta^* > 0$ such that for $\delta > \delta^* > 0$ the profile equation* (2.63) *admits a strictly increasing solution satisfying*

$$\lim_{s\to-\infty} \psi_2(s) = 0 \quad \textit{and} \quad \lim_{s\to\infty} \psi_2(s) = 1. \tag{2.64}$$

In fact, the condition $V_p + V_v < x_2^+$ is equivalent to the condition

$$\int_0^1 f_2(x)\,dx = \frac{\beta(V_v - V_p)(V_v + V_p - x_2^+)}{x_2^+ 2} > 0 \tag{2.65}$$

which is required in [19] when f_2 is a Lipschitz continuous function.

Theorem 2.7 shows the existence of strictly increasing traveling wave solutions connecting two nonneighboring equilibrium solutions 0 and 1. In contrast, it seems difficult to obtain similar results by using the techniques of monotone iteration method due to the lack of appropriate upper–lower solution pairs that one can construct. This issue has become the subject of current research of the authors.

3. Monotonic standing wave solutions ($c = 0$)

Observe that for the same templates $(a, \gamma, \delta) \in \Omega_1$ there exist monotonic nondecreasing solutions of (1.5) with (BC1) for $c_1 < c < 0$ and $c > 0$. Similar situations occur in cases (1-2) and (1-6), cases (2-1) and (2-5), and cases (2-2) and (2-6) in Theorem 1. Motivated by these basic observations, in this section, we are going to study the failure of wave propagation ($c = 0$). In other words, we attempt looking for standing wave solutions of (1.3), or equivalently, seeking $\{\varphi(i)\}_{i\in\mathbf{Z}}$ satisfying (1.6).

To this end, we will establish a discrete version of the monotone iteration scheme, and then use this scheme to prove the existence of monotonic standing wave solutions with various boundary conditions for (γ,δ) in Ω_1 or Ω_2.

3.1. Construction of upper and lower solutions

The underlying idea is similar to the continuous case. We say that the sequences $\{\bar{U}(i)\mid i\in\mathbf{Z}\}$ and $\{\bar{L}(i)\mid i\in\mathbf{Z}\}$ are, respectively, upper and lower solutions of (1.6) if

$$0\geqslant -g(\bar{U}(i))+a\bar{U}(i)+\gamma\bar{U}(i-1)+\delta\bar{U}(i+1), \tag{3.1}$$

$$0\leqslant -g(\bar{L}(i))+a\bar{L}(i)+\gamma\bar{L}(i-1)+\delta\bar{L}(i+1), \tag{3.2}$$

for all $i\in\mathbf{Z}$. Let Ξ denote the space $\{\psi\mid\psi:\ \mathbf{Z}\to[x_0,x_2^+]\}$. Define the following standing profile spaces Ξ_i, for $i=1,2,3,4$:

$$\Xi_1=\{\psi\in\Xi\mid\psi(i)\in[x_0,x_1^+],\psi(i)\leqslant\psi(i+1)\ \text{for}\ i\in\mathbf{Z},\ \text{and (BC1) holds}\},$$

$$\Xi_2=\{\psi\in\Xi\mid\psi(i)\in[x_0,x_1^+],\psi(i+1)\leqslant\psi(i)\ \text{for}\ i\in\mathbf{Z},\ \text{and (BC2) holds}\},$$

$$\Xi_3=\{\psi\in\Xi\mid\psi(i)\in[x_1^+,x_2^+],\psi(i)\leqslant\psi(i+1)\ \text{for}\ i\in\mathbf{Z},\ \text{and (BC3) holds}\},$$

$$\Xi_4=\{\psi\in\Xi\mid\psi(i)\in[x_1^+,x_2^+],\psi(i+1)\leqslant\psi(i)\ \text{for}\ i\in\mathbf{Z},\ \text{and (BC4) holds}\}.$$

Next, we define the monotone operator D on Ξ by

$$D(\psi)(i)=\frac{1}{\mu}(-g(\psi(i))+(a+\mu)\psi(i)+\gamma\psi(i-1)+\delta\psi(i+1)), \tag{3.3}$$

where $\mu>0$ is sufficiently large such that $\mu+a-\alpha>0$. It is ready to show that a function $\varphi\in\Xi$ is a solution of (1.6) if and only if φ is a fixed point of D. Moreover, some properties of the operator D can be stated as follows:

Lemma 3.1. *Assume that* $\psi,\tilde{\psi}\in\Xi$.

(1) ψ *is an upper (resp., a lower) solution of* (1.6) *if and only if* $\psi(i)\geqslant$ (*resp.*, $\leqslant$) $D(\psi)(i)$ *for all* $i\in\mathbf{Z}$.
(2) *If* $\psi(i)\leqslant\tilde{\psi}(i)$ *for all* $i\in\mathbf{Z}$, *then* $D(\psi)(i)\leqslant D(\tilde{\psi})(i)$ *for all* $i\in\mathbf{Z}$.
(3) *If* ψ *is an upper (resp., a lower) solution of* (1.6), *then* $D(\psi)$ *is also an upper (resp., a lower) solution of* (1.6).
(4) *If* $\psi\in\Xi_j$, *then* $D(\psi)\in\Xi_j$, *too.*

Proof.

(1) Suppose ψ is an upper (resp., a lower) solution of (1.6). By (3.1) (resp., (3.2)), we have

$$\begin{aligned}\mu\psi(i) &\geqslant (\text{resp.,} \leqslant) - g(\psi(i)) + (a+\mu)\psi(i) + \gamma\psi(i-1) + \delta\psi(i+1),\\ &= \mu D(\psi)(i), \quad \text{for all } i \in \mathbf{Z}. \end{aligned} \tag{3.4}$$

On the other hand, if $\psi(i) \geqslant$ (resp., $\leqslant$) $D(\psi)(i)$ for all $i \in \mathbf{Z}$ but ψ is not an upper (resp., a lower) solution of (1.6), then there must exist some integer k such that

$$0 < (\text{resp.,} >) - g(\psi(k)) + a\psi(k) + \gamma\psi(k-1) + \delta\psi(k+1) \tag{3.5}$$

which implies that $\psi(k) < (\text{resp.,} >)$ $D(\psi)(k)$. This leads to a contradiction.

(2) The assertion is evidently true due to the choice of μ.

(3) Assume ψ is an upper (resp., a lower) solution of (1.6). Combining (1) with (2), we have

$$\begin{aligned}&\psi(i) \geqslant D(\psi)(i) \geqslant D^{(2)}(\psi)(i),\\ &(\text{resp.,}\ \psi(i) \leqslant D(\psi)(i) \leqslant D^{(2)}(\psi)(i)),\end{aligned} \tag{3.6}$$

for all $i \in \mathbf{Z}$. Hence, by (1) again, $D(\psi)$ is also an upper (resp., a lower) solution of (1.6).

(4) For simplicity, we only prove the case for $\psi \in \Xi_1$, and the results for other cases can be done in a similar way. Since $\psi(i) \leqslant \psi(i+1)$ for all $i \in \mathbf{Z}$, we have

$$\begin{aligned} D(\psi)(i) &= \frac{1}{\mu}(-g(\psi(i)) + (a+\mu)\psi(i) + \gamma\psi(i-1) + \delta\psi(i+1))\\ &\leqslant \frac{1}{\mu}(-g(\psi(i+1)) + (a+\mu)\psi(i+1) + \gamma\psi(i) + \delta\psi(i+2))\\ &= D(\psi)(i+1), \end{aligned}$$

for all $i \in \mathbf{Z}$ which implies $D(\psi)$ is monotonic nondecreasing on $\mathbf{Z}$. Furthermore, a simple computation shows that

$$\lim_{i\to-\infty} D(\psi)(i) = x_0 \quad \text{and} \quad \lim_{i\to\infty} D(\psi)(i) = x_1^+.$$

Hence $D(\psi) \in \Xi_1$.

This completes the proof. $\square$

We are now in the position to construct the practical upper and lower solutions of (1.6) for $(\gamma, \delta) \in \Omega_1$ (or Ω_2) through the help of the roots of characteristic function. If

$(\gamma, \delta) \in \Omega_1$ (resp., Ω_2) then there exists $\lambda^- < 0$ (resp., $0 < \lambda^+$) satisfying

$$\triangle(\lambda^-, 0; x_1^+) = \alpha + 2\beta - a - \gamma e^{-\lambda^-} - \delta e^{\lambda^-} = 0, \tag{3.7}$$

$$\triangle(\lambda^- - \varepsilon, 0; x_1^+) > 0, \quad \text{for all sufficiently small } \varepsilon > 0 \tag{3.8}$$

(resp.,

$$\triangle(\lambda^+, 0; x_1^+) = \alpha + 2\beta - a - \gamma e^{-\lambda^+} - \delta e^{\lambda^+} = 0,$$

$$\triangle(\lambda^+ + \varepsilon, 0; x_1^+) > 0, \quad \text{for all sufficiently small } \varepsilon > 0).$$

Combining these results with Proposition 2.4 (see also, Lemma 2.1), we can define the following four upper–lower solution pairs of (1.6):

Lemma 3.2. *Assume that* (1.7) *holds.*

(1) *Let* $(\gamma, \delta) \in \Omega_1$. *We replace the negative number* σ_1 *by* λ^- *in the formulas of* $(U_1(s), L_1(s))$ *and* $(U_2(s), L_2(s))$ *in Table* 5, *and then define* $\bar{U}_j$, $\bar{L}_j \in \Xi$ *for* $j = 1, 2$ *by*

$$(\bar{U}_1(i), \bar{L}_1(i)) = (U_1(i), L_1(i)), \quad \textit{for all } i \in \mathbf{Z}, \tag{3.9}$$

$$(\bar{U}_2(i), \bar{L}_2(i)) = (U_2(i), L_2(i)), \quad \textit{for all } i \in \mathbf{Z}. \tag{3.10}$$

If $\zeta > 0$ *is small enough, then* $(\bar{U}_1, \bar{L}_1)$ *and* $(\bar{U}_2, \bar{L}_2)$ *are upper–lower solution pairs of* (1.6).

(2) *Let* $(\gamma, \delta) \in \Omega_2$. *We replace the positive number* σ_7 *by* λ^+ *in the formulas of* $(U_7(s), L_7(s))$ *and* $(U_8(s), L_8(s))$ *in Table* 6, *and then define* $\bar{U}_j$, $\bar{L}_j \in \Xi$ *for* $j = 3, 4$ *by*

$$(\bar{U}_3(i), \bar{L}_3(i)) = (U_7(i), L_7(i)), \quad \textit{for all } i \in \mathbf{Z}, \tag{3.11}$$

$$(\bar{U}_4(i), \bar{L}_4(i)) = (U_8(i), L_8(i)), \quad \textit{for all } i \in \mathbf{Z}. \tag{3.12}$$

If $\zeta > 0$ *is small enough, then* $(\bar{U}_7, \bar{L}_7)$ *and* $(\bar{U}_8, \bar{L}_8)$ *are upper–lower solution pairs of* (1.6).

Proof. Note that if we replace σ_1 by λ^- in the formulas of $(U_1(s), L_1(s))$ and $(U_2(s), L_2(s))$ in Table 5 then, for sufficiently small $\zeta > 0$, $(U_1(s), L_1(s))$ and $(U_2(s), L_2(s))$ are still upper–lower solution pairs of (1.5) with $c = 0$. Hence, both $(\bar{U}_1, \bar{L}_1)$ and $(\bar{U}_2, \bar{L}_2)$ are upper–lower solution pairs of (1.6). Part (2) can be proved in a similar way. This completes the proof. $\square$

3.2. A discrete monotone iteration scheme for the existence of monotonic standing wave solutions

In this subsection, with the aid of upper and lower solutions constructed above, we will prove the results of part (1) and part (2) in Theorem 2 by using a discrete monotone iteration scheme.

Proof of parts (1) and (2) of Theorem 2. For simplicity, we only prove the existence of monotonic nondecreasing standing wave solution satisfying (BC1) in part (1). The remaining parts can be obtained by the similar arguments.

Since $\bar{L}_1$ is a lower solution in Ξ_1, according to Lemma 3.1, we know that $D^{(n)}(\bar{L}_1)$, for $n = 1, 2, \dots$, are still lower solutions in Ξ_1 and, for each $i \in \mathbf{Z}$,

$$x_0 \leqslant \bar{L}_1(i) \leqslant D^{(1)}(\bar{L}_1)(i) \leqslant \cdots \leqslant D^{(n)}(\bar{L}_1)(i) \leqslant \cdots \leqslant \bar{U}_1(i) \leqslant x_1^+.$$

Therefore, for each $i \in \mathbf{Z}$, $\{D^{(n)}(\bar{L}_1)(i)\}_{n=0}^{\infty}$ is a monotonic nondecreasing and bounded sequence which implies that there exists a sequence of real numbers $\{v_i\}_{i=-\infty}^{\infty}$ such that

$$\lim_{n \to \infty} D^{(n)}(\bar{L}_1)(i) = v_i, \quad \text{for all } i \in \mathbf{Z}. \tag{3.13}$$

Since $D^{(n)}(\bar{L}_1) \in \Xi_1$ for all $n \geqslant 0$, one can prove that $\{v_i\}_{i=-\infty}^{\infty}$ is also monotonic nondecreasing. Define the function $\varphi : \mathbf{Z} \to [x_0, x_1^+]$ by $\varphi(i) = v_i$. Then for each $i \in \mathbf{Z}$, we have

$$\varphi(i) = \lim_{n \to \infty} D^{(n+1)}(\bar{L}_1)(i) = \lim_{n \to \infty} D(D^{(n)}(\bar{L}_1))(i) = D(\varphi)(i).$$

Hence, φ is a fixed point of D. Finally, using $\bar{U}_1$ as the barrier function, one can verify that $\lim_{i \to -\infty} \varphi(i) = x_0$ and $\lim_{i \to \infty} \varphi(i) = x_1^+$. This completes the proof. □

Observe that if $\gamma = 0$ or $\delta = 0$ then the profile equation (1.6) can be viewed as an one-dimensional iteration map problem. Thus, in what follows, we will explore the multiplicity of monotonic standing wave solutions of (1.3) for $(\gamma, \delta) \in \Omega_1$ with $\gamma = 0$ and $(\gamma, \delta) \in \Omega_2$ with $\delta = 0$ by the techniques of dynamical systems for maps.

3.3. Monotonic standing waves of advanced type ($\gamma = 0$)

In this case, the profile equation (1.6) is changed into

$$\varphi(i+1) = \frac{g(\varphi(i)) - a\varphi(i)}{\delta}, \quad \text{for all } i \in \mathbf{Z}. \tag{3.14}$$

Thus, if $(\gamma, \delta) \in \Omega_1$ with $\gamma = 0$ then $\lambda^- < 0$ satisfies

$$\triangle(\lambda^-, 0; x_1^+) = \alpha + 2\beta - a - \delta e^{\lambda^-} = 0. \tag{3.15}$$

Hence, $\lambda^- = \ln\{(\alpha + 2\beta - a)/\delta\}$. Corresponding to part (1) of Theorem 2, we are interested in studying the multiplicity of monotonic nondecreasing solutions of (3.14) with (BC1) and monotonic nonincreasing solutions of (3.14) with (BC4).

3.3.1. Nondecreasing standing waves with (BC1)

First of all, noting that $g(x) = \alpha x$ for $0 \leqslant x \leqslant V_p$ and then combining this fact with an observation on (3.14), we define the following one-to-one map $Q_1 : \mathbf{R} \to \mathbf{R}$ by

$$Q_1(x) = \frac{\alpha x - ax}{\delta}. \tag{3.16}$$

The graph of the map Q_1 is shown in Fig. 5. Let $x_i = \varphi(i)$ for all $i \in \mathbf{Z}$. If $0 \leqslant x_i \leqslant V_p$ then (3.14) can be written as an one-dimensional iteration map problem by

$$x_{i+1} = Q_1(x_i). \tag{3.17}$$

Note that, by (1.7), we have

$$Q_1{}'(x) = \frac{\alpha - a}{\delta} > 1. \tag{3.18}$$

Let x_L, x_p, and x_R be the images of Q_1 at 0, V_p and x_1^+, respectively, from (3.16) and (3.18), we immediately conclude that

$$x_L = 0 < x_p = \frac{(\alpha - a)V_p}{\delta} < x_R = \frac{(\alpha - a)x_1^+}{\delta}. \tag{3.19}$$

Since $Q_1(0) = x_L = 0$, x_L is a repelling fixed point of Q_1.

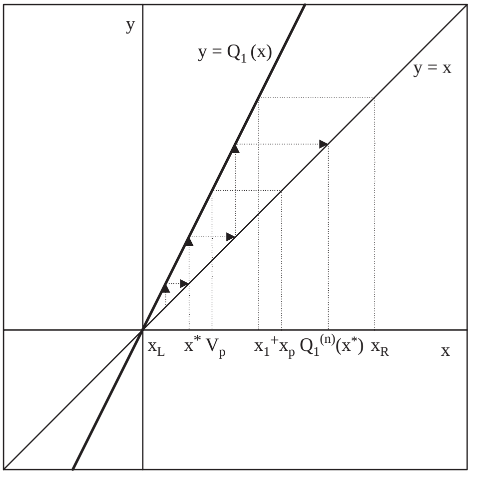


Fig. 5. The graph of iteration map Q_1.

Next, define the interval $I_1 = [x_p, x_R]$ and the nonempty set Λ_1 by

$$\Lambda_1 = \bigcup_{j=-1}^{-\infty} Q_1^{(j)}(I_1). \tag{3.20}$$

Now, for an arbitrarily chosen $x^* \in \Lambda_1$, there exists a positive integer $n = n(x^*)$ such that $Q_1^{(n)}(x^*) \in I_1$, but $Q_1^{(n-1)}(x^*) \notin I_1$ with

$$0 \leqslant Q_1^{-1}(Q_1^{(n-1)}(x^*)) < V_p \leqslant Q_1^{-1}(Q_1^{(n)}(x^*)) \leqslant x_1^+. \tag{3.21}$$

Define a function $\varphi: \mathbf{Z} \to [0, x_1^+]$ by $\varphi(n) = Q_1^{-1}(Q_1^{(n)}(x^*))$ and

$$\varphi(i) = \begin{cases} x_1^+ + (\varphi(n) - x_1^+)e^{(i-n)\lambda^-} & \text{for } i > n, \\ Q_1^{-1}(Q_1^{(i)}(x^*)) & \text{for } i < n. \end{cases} \tag{3.22}$$

Then one can check that φ is a monotonic nondecreasing solution of (3.14) with (BC1). Since the nonempty subset $Q_1^{-1}(I_1) \subset \Lambda_1$ has positive measure and x^* is arbitrarily chosen from Λ_1, we can conclude that there are infinitely many monotonic nondecreasing solutions of (3.14) satisfying (BC1).

3.3.2. Nonincreasing standing waves with (BC4)

Our idea is similar to that in Section 3.3.1. Motivated by the observation that $g(x) = \alpha x + 2m$ for $x \geqslant V_v$, we define the following one-to-one map $Q_2: \mathbf{R} \to \mathbf{R}$ by

$$Q_2(x) = \frac{\alpha x + 2m - ax}{\delta}. \tag{3.23}$$

See the graph of Q_2 as shown in Fig. 6. Let $x_i = \varphi(i)$ for all $i \in \mathbf{Z}$. If $x_i \geqslant V_v$ then (3.14) can be written as an one-dimensional iteration map problem by

$$x_{i+1} = Q_2(x_i). \tag{3.24}$$

Let x_L, x_v, and x_R be the images of Q_2 at x_1^+, V_v, and x_2^+, respectively. Then by (3.23) and the fact that

$$Q_2'(x) = \frac{\alpha - a}{\delta} > 1, \tag{3.25}$$

we have

$$x_L = \frac{(\alpha - a)x_1^+ + 2m}{\delta} < x_v = \frac{(\alpha - a)V_v + 2m}{\delta} < x_R = x_2^+. \tag{3.26}$$

Since $Q_2(x_2^+) = x_R = x_2^+$, x_R is a repelling fixed point of Q_2.

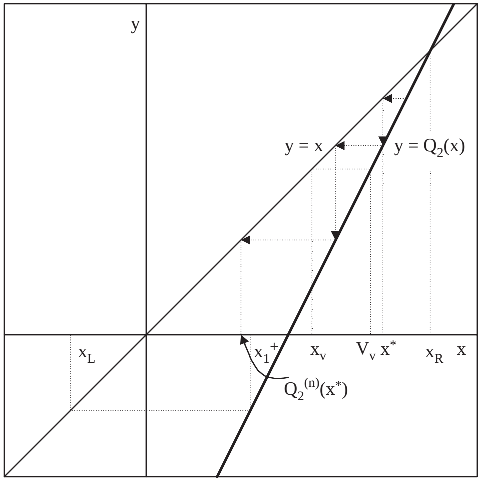


Fig. 6. The graph of iteration map Q_2.

Define the interval $I_2 = [x_L, x_v]$ and the nonempty set Λ_2 by

$$\Lambda_2 = \bigcup_{j=-1}^{-\infty} Q_2^{(j)}(I_2). \tag{3.27}$$

Then for an arbitrarily chosen $x^* \in \Lambda_2$, there exists a positive integer $n = n(x^*)$ such that $Q_2^{(n)}(x^*) \in I_2$, but $Q_2^{(n-1)}(x^*) \notin I_2$ with

$$x_1^+ \leqslant Q_2^{-1}(Q_2^{(n)}(x^*)) \leqslant V_v < Q_1^{-1}(Q_1^{(n-1)}(x^*)) \leqslant x_2^+. \tag{3.28}$$

Define a function $\varphi :\ \mathbf{Z} \to [x_1^+, x_2^+]$ by $\varphi(n) = Q_2^{-1}(Q_2^{(n)}(x^*))$ and

$$\varphi(i) = \begin{cases} x_1^+ + (\varphi(n) - x_1^+)e^{(i-n)\lambda^-} & \text{for } i > n, \\ Q_2^{-1}(Q_2^{(i)}(x^*)) & \text{for } i < n. \end{cases} \tag{3.29}$$

Then φ is a monotonic nonincreasing solution of (3.14) with (BC4) and, since the nonempty subset $Q_2^{-1}(I_2) \subset \Lambda_2$ has positive measure, we actually proved that there are infinitely many monotonic nonincreasing solutions of (3.14) satisfying (BC4). This completes the proof of part (3) of Theorem 2.

3.4. *Monotonic standing waves of delayed type* ($\delta = 0$)

For $(\gamma, \delta) \in \Omega_2$ with $\delta = 0$, the profile equation (1.6) is changed into

$$-g(\varphi(i)) + a\varphi(i) + \gamma\varphi(i-1) = 0, \quad \text{for all } i \in \mathbf{Z}, \tag{3.30}$$

and the associated characteristic function is

$$\triangle(\sigma, 0; x_1^+) = \alpha + 2\beta - a - \gamma e^{-\sigma}. \tag{3.31}$$

Thus, the positive root λ^+ of the characteristic function (3.31) can be found in an explicit form, $\lambda^+ = \ln\{\gamma/(\alpha + 2\beta - a)\}$. Now, corresponding to part (2) of Theorem 2, we are interested in studying the multiplicity of monotonic nonincreasing solutions of (3.30) with (BC2) and monotonic nondecreasing solutions of (3.30) with (BC3).

3.4.1. Nonincreasing standing waves with (BC2)

Motivated by that $g(x) = \alpha x$ for $0 \leqslant x \leqslant V_p$, we first define the one-to-one map $f_1 : \mathbf{R} \to \mathbf{R}$ by

$$f_1(x) = -\alpha x + ax. \tag{3.32}$$

Let $y = f_1(x)$. Then we have

$$x = \frac{y}{a - \alpha}.$$

Denote $x_i = \varphi(i)$ and $y_i = f_1(x_i)$ for all $i \in \mathbf{Z}$. If $0 \leqslant x_i \leqslant V_p$ then (3.30) can be written as

$$y_i + \gamma x_{i-1} = 0 \tag{3.33}$$

which implies that

$$y_i = -\gamma x_{i-1} = \frac{\gamma y_{i-1}}{\alpha - a}. \tag{3.34}$$

Thus, we define another one-to-one map $F_1 : \mathbf{R} \to \mathbf{R}$ by

$$F_1(y) = \frac{\gamma y}{\alpha - a}. \tag{3.35}$$

The graph of F_1 is shown in Fig. 7. We remark that if $0 \leqslant y_i/(a - \alpha) \leqslant V_p$, then (3.30) can be written as an one-dimensional iteration map problem by

$$y_i = F_1(y_{i-1}) = \frac{\gamma y_{i-1}}{\alpha - a}. \tag{3.36}$$

Let y_L, y_p and y_R be the images of f_1 at x_1^+, V_p and 0, respectively. Since f_1 is a strictly decreasing function, we have

$$y_L = (a - \alpha)x_1^+ < y_p = (a - \alpha)V_p < y_R = 0. \tag{3.37}$$

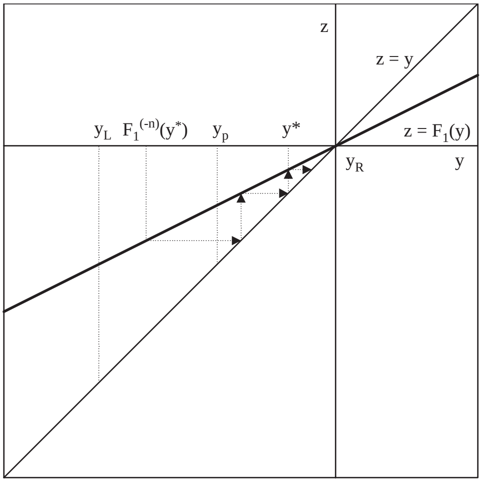


Fig. 7. The graph of iteration map F_1.

Note that, by (1.7),

$$0 < F_1{}'(y) = \frac{\gamma}{\alpha - a} < 1 \quad \text{and} \quad F_1(y_R) = 0 = y_R. \tag{3.38}$$

Hence y_R is an attracting fixed point of F_1.

Define the interval $J_1 = [y_L, y_p]$ and set the nonempty set Σ_1 by

$$\Sigma_1 = \bigcup_{i=1}^{\infty} F_1^{(i)}(J_1). \tag{3.39}$$

For an arbitrary chosen $y^* \in \Sigma_1$, there exists a positive integer $n = n(y^*)$ such that $F_1^{(-n)}(y^*) \in J_1$ and $F_1^{(-n+1)}(y^*) \notin J_1$ with

$$0 \leqslant f_1^{-1}(F_1^{(-n+1)}(y^*)) < V_p \leqslant f_1^{-1}(F_1^{(-n)}(y^*)) \leqslant x_1^+. \tag{3.40}$$

Define a function $\varphi: \mathbf{Z} \to [0, x_1^+]$ by $\varphi(n) = f_1^{-1}(F_1^{(-n)}(y^*))$ and

$$\varphi(i) = \begin{cases} x_1^+ + (\varphi(n) - x_1^+)e^{(i-n)\lambda^+} & \text{for } i < n, \\ f_1^{-1}(F_1^{(i-2n)}(y^*)) & \text{for } i > n. \end{cases} \tag{3.41}$$

Then φ is a monotonic nonincreasing solution of (3.30) with (BC2) and, since the nonempty subset $F_1^{(1)}(J_1) \subset \Sigma_1$ has positive measure, we actually proved that there are infinitely many monotonic nonincreasing solutions of (3.30) satisfying (BC2).

3.4.2. Nondecreasing standing waves with (BC3)

The idea is similar to that in Section 3.4.1. We first define two one-to-one maps, $f_2: \mathbf{R} \to \mathbf{R}$ and $F_2: \mathbf{R} \to \mathbf{R}$, by

$$f_2(x) = -\alpha x - 2m + ax, \tag{3.42}$$

$$F_2(y) = \gamma\left(\frac{y+2m}{\alpha - a}\right), \tag{3.43}$$

where $y = f_2(x)$. Let $x_i = \varphi(i)$ and $y_i = f_2(x_i)$ for all $i \in \mathbf{Z}$. If $(y_i + 2m)/(a-\alpha) \geqslant V_v$ then (3.30) can be written as an one-dimensional iteration map problem by

$$y_i = F_2(y_{i-1}) = \gamma\left(\frac{y_{i-1}+2m}{\alpha - a}\right). \tag{3.44}$$

The graph of F_2 is shown in Fig. 8.

Denote the images of the strictly decreasing function f_2 at x_2^+, V_v, and x_1^+ by y_L, y_v, and y_R, respectively. Then we have

$$y_L = (a-\alpha)x_2^+ - 2m < y_v = (a-\alpha)V_v - 2m < y_R = (a-\alpha)x_1^+ - 2m. \tag{3.45}$$

By simple computation, we obtain

$$0 < F_2{}'(y) = \frac{\gamma}{\alpha - a} < 1 \quad \text{and} \quad F_2(y_L) = y_L < 0. \tag{3.46}$$

Hence y_L is an attracting fixed point of F_2.

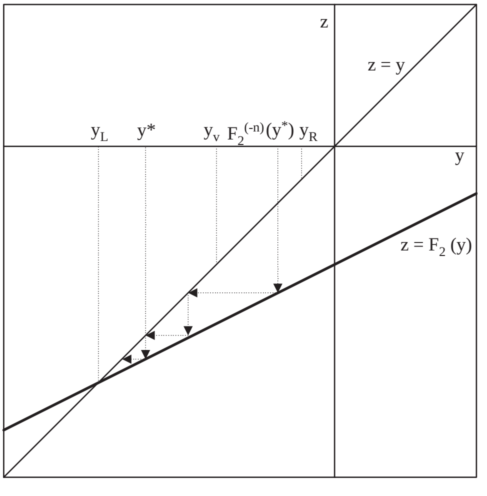


Fig. 8. The graph of iteration map F_2.

Define the interval $J_2 = [y_v, y_R]$ and the nonempty set Σ_2 by

$$\Sigma_2 = \bigcup_{i=1}^{\infty} F_2^{(i)}(J_2). \tag{3.47}$$

Now, choosing an arbitrary $y^* \in \Sigma_2$, one can conclude that there exists a positive integer $n = n(y^*)$ such that $F_2^{(-n)}(y^*) \in J_2$ and $F_2^{(-n+1)}(y^*) \notin J_2$ with

$$x_1^+ \leqslant f_2^{-1}(F_2^{(-n)}(y^*)) \leqslant V_v < f_2^{-1}(F_2^{(-n+1)}(y^*)). \tag{3.48}$$

Define a function $\varphi: \mathbf{Z} \to [x_1^+, x_2^+]$ by $\varphi(n) = f_2^{-1}(F_2^{(-n)}(y^*))$ and

$$\varphi(i) = \begin{cases} x_1^+ + (\varphi(n) - x_1^+)e^{(i-n)\lambda^+} & \text{for } i < n, \\ f_2^{-1}(F_2^{(i-2n)}(y^*)) & \text{for } i > n. \end{cases} \tag{3.49}$$

Then one can verify that φ is a monotonic nondecreasing solution of (3.30) satisfying (BC3) and, since the nonempty subset $F_2^{(1)}(J_2) \subset \Sigma_2$ has positive measure, we have proved that there are infinitely many monotonic nondecreasing solutions of (3.30) with (BC3). This completes the proof of part (4) of Theorem 2.

4. Numerical simulation

In this section we focus on the numerical simulation of the monotone iteration scheme for the existence of traveling wave solutions. Using case (1-1) in Theorem 1 as an example, we can simulate the deformation of the lower solutions by solving a sequence of initial-value problems as follows.

First, recall the initial lower solution $L_1^{(0)}(s) = L_1(s)$ of case (1-1), which is given by

$$L_1^{(0)}(s) = \begin{cases} x_1^+ - x_1^+ e^{\sigma_1 s}, & s \geqslant 0, \\ 0, & s \leqslant 0. \end{cases} \tag{4.1}$$

Then the sequence of lower solutions $\{L_1^{(n)}(s)\}$ is generated by

$$L_1^{(n+1)}(s) = T_1(L_1^{(n)})(s) = e^{-\mu_1 s} \int_{-\infty}^{s} e^{\mu_1 t} H_1(L_1^{(n)})(t)\, dt, \tag{4.2}$$

for all $n \geqslant 0$, where parameter $\mu_1 > 0$ is chosen to be sufficiently large. Indeed, we take $\mu_1 = (-1/c)(\alpha - a) + 100$ in our numerical experiment. Observe that

$$L_1^{(n+1)}(s) = 0, \quad \text{for } s \leqslant -(n+1) \text{ and for all } n \geqslant 0. \tag{4.3}$$

Now differentiating (4.2), we find that $L_1^{(n+1)}(s)$ satisfies the following differential equation:

$$\frac{dL_1^{(n+1)}(s)}{ds} + \mu_1 L_1^{(n+1)}(s) = H_1(L_1^{(n)})(s). \tag{4.4}$$

Thus we have induced a sequence of initial-value problems, (4.4) with (4.3) for all $n \geqslant 0$.

Next, we solve the initial-value problem (4.3) with (4.4) by some numerical solver. For example, we adopt a second-order Runge–Kutta method (the modified Euler scheme) with small mesh size $\Delta s = 10^{-3}$.

With a close inspection, one can find that the similar ideas can be applied to cases (1-2), (2-1), and (2-2) in Theorem 1. However, since the wave speed c is positive in cases (2-1) and (2-2), we choose the parameters $\mu_i = (-1/c)(a-\alpha) + 100$, for $i = 2, 4$, in our numerical experiments. The following four examples demonstrate the numerical simulation.

Example 4.1 (Case 1-1). We first take the template $(a, \gamma, \delta) = (-2.0, 0.01, 2.0)$. Choosing the parameters in the piecewise-linear function g by $\alpha = 0.5001$, $\beta = -1.0$, $V_p = 1.5$, and $V_v = 2.0$, one can check that $(\gamma, \delta) \in \Omega_1$, and the three nonnegative equilibrium solutions to the profile equation (1.5) are

$$x_0 = 0.0, \quad x_1^+ = 1.98688654877806, \quad \text{and} \quad x_2^+ = 2.04039991838400.$$

Taking the wave speed $c = -0.05$, we then use the bisection method to find the two negative roots λ_1 and λ_2 of the corresponding characteristic function $\triangle(\sigma, -0.05; x_1^+)$:

$$\lambda_1 = -1.71951495390216 \quad \text{and} \quad \lambda_2 = -3.25926296378180,$$

where λ_1 plays the role σ_1 in Lemma 2.1. Finally, we solve the initial-value problem (4.4) with (4.3) for $n = 0, 1, \ldots, 34$, where $\mu_1 = 150.002$. The deformation of the lower solutions can be observed in Fig. 9. Numerical evidence shows that the sequence of lower solutions approaches to a limiting function which is the monotonic nondecreasing traveling wave solution satisfying (BC1).

Example 4.2 (Case 1-2). For simplicity, we take the same parameters as that in Example 4.1, but consider the monotonic nonincreasing traveling wave solution connecting x_2^+ with x_1^+. We solve the following sequence of initial-value problems numerically:

$$\begin{cases} \dfrac{dU_2^{(n+1)}(s)}{ds} + \mu_3 U_2^{(n+1)}(s) = H_3(U_2^{(n)})(s), \\ U_2^{(n+1)}(s) = x_2^+ \quad \text{for } s \leqslant -(n+1) \text{ and for all } n \geqslant 0, \end{cases} \tag{4.5}$$

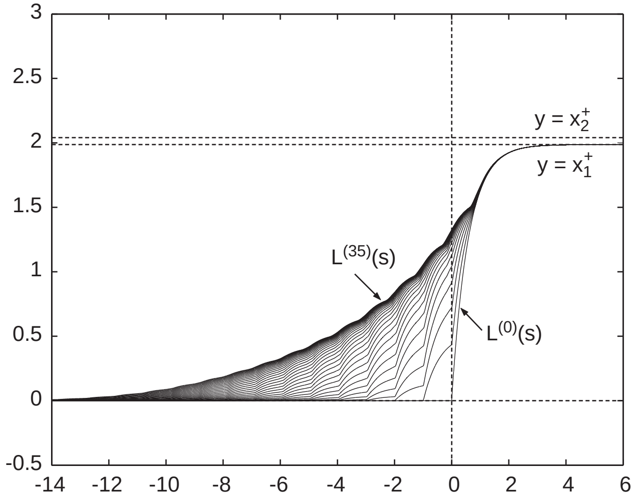


Fig. 9. Deformation of lower solutions in Example 4.1.

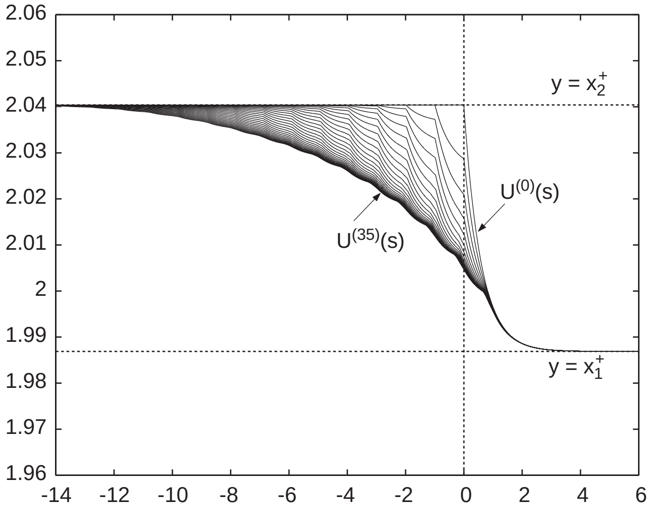


Fig. 10. Deformation of upper solutions in Example 4.2.

where $\mu_3 = 150.002$ and the initial upper solution $U_2^{(0)}(s) = U_2(s)$ is given in Table 5 by

$$U_2^{(0)}(s) := \begin{cases} x_1^+ + (x_2^+ - x_1^+)e^{\sigma_1 s}, & s \geqslant 0, \\ x_2^+, & s \leqslant 0. \end{cases} \tag{4.6}$$

The deformation of the upper solutions is shown in Fig. 10.

Example 4.3 (Case 2-1). We change the template to $(a,\gamma,\delta)=(-2.0,2.0,0.01)$. Choosing the parameters in the piecewise-linear function g the same with Example 4.1, we can check that $(\gamma,\delta)\in\Omega_2$, and the three nonnegative equilibrium solutions to the profile equation (1.5) are still

$$x_0 = 0.0, \quad x_1^+ = 1.98688654877806, \quad \text{and} \quad x_2^+ = 2.04039991838400.$$

Taking the positive wave speed $c=0.05$, we use the bisection method to find the two positive roots λ_1 and λ_2 of the corresponding characteristic function $\triangle(\sigma,0.05;x_1^+)$:

$$\lambda_1 = 1.71951495390216 \quad \text{and} \quad \lambda_2 = 3.25926296378180,$$

where λ_1 plays the role σ_7 in Lemma 2.1. Similarly, we solve the following sequence of initial-value problems:

$$\begin{cases} \dfrac{dL_7^{(n+1)}(s)}{ds} - \mu_2 L_7^{(n+1)}(s) = H_2(L_7^{(n)})(s), \\ L_7^{(n+1)}(s) = 0, \quad \text{for } s\geqslant(n+1) \text{ and for all } n\geqslant 0, \end{cases} \tag{4.7}$$

where $\mu_2 = 150.002$ and the initial lower solution $L_7^{(0)}(s) = L_7(s)$ is given in Table 6 by

$$L_7^{(0)}(s) := \begin{cases} 0, & s\geqslant 0, \\ x_1^+ - x_1^+ e^{\sigma_7 s}, & s\leqslant 0. \end{cases} \tag{4.8}$$

The deformation of the lower solutions is shown in Fig. 11. Numerical evidence shows that the sequence of lower solutions approaches to a limiting function which is the monotonic nonincreasing traveling wave solution connecting x_1^+ and x_0.

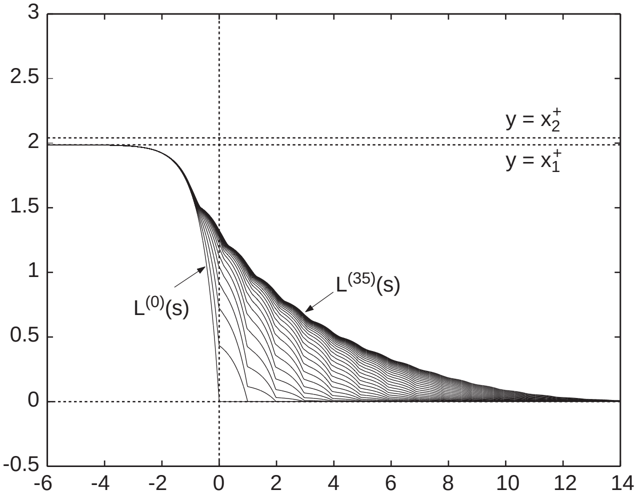


Fig. 11. Deformation of lower solutions in Example 4.3.

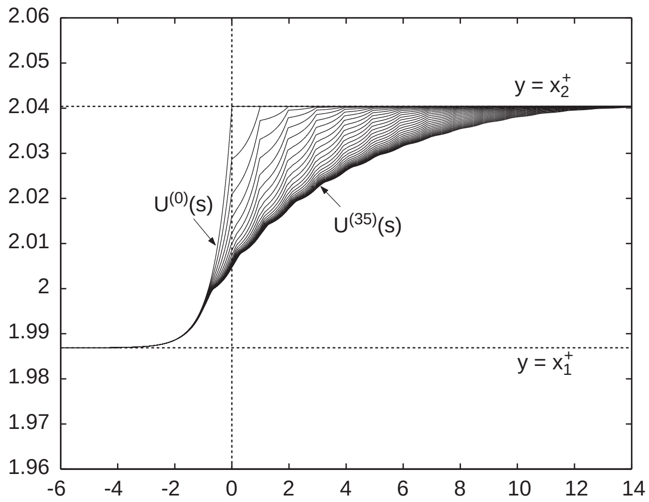


Fig. 12. Deformation of upper solutions in Example 4.4.

Example 4.4 (Case 2-2). Again, for simplicity, we take the same parameters as that in Example 4.3, but consider the monotonic nondecreasing traveling wave solution connecting x_1^+ with x_2^+. We seek the approximations to the following sequence of initial-value problems:

$$\begin{cases} \dfrac{dU_8^{(n+1)}(s)}{ds} - \mu_4 U_8^{(n+1)}(s) = H_4(U_8^{(n)})(s), \\ U_8^{(n+1)}(s) = x_2^+ \quad \text{for } s \geqslant (n+1) \text{ and for all } n \geqslant 0, \end{cases} \tag{4.9}$$

where $\mu_4 = 150.002$ and the initial upper solution $U_8^{(0)}(s) = U_8(s)$ is given in Table 6 by

$$U_8^{(0)}(s) := \begin{cases} x_2^+, & s \geqslant 0, \\ x_1^+ + (x_2^+ - x_1^+)e^{\sigma_7 s}, & s \leqslant 0. \end{cases} \tag{4.10}$$

The deformation of the upper solutions is shown in Fig. 12.

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References

[1] S.-N. Chow, J. Mallet-Paret, W. Shen, Traveling waves in lattice dynamical systems, J. Differential Equations 149 (1998) 248–291.

[2] R. Dogaru, L.O. Chua, M. Hänggi, A compact and universal RTD-based CNN cell: circuit, piecewise-linear model and functional capabilities, Memorandum UCB/ERL M99/72, Electronic Research Laboratory, University of California, Berkeley, 1999.

[3] T. Erneux, G. Nicolis, Propagation waves in discrete bistable reaction-diffusion systems, Physica D 67 (1993) 237–244.

[4] G. Fáth, Propagation failure of traveling waves in a discrete bistable medium, Physica D 116 (1998) 176–190.

[5] M. Hänggi, L.O. Chua, Simulation of RTD-based CNN cells, Memorandum UCB/ERL M00/51, Electronic Research Laboratory, University of California, Berkeley, 2000.

[6] C.-H. Hsu, S.-S. Lin, Existence and multiplicity of traveling waves in a lattice dynamical system, J. Differential Equations 164 (2000) 431–450.

[7] C.-H. Hsu, S.-S. Lin, W. Shen, Traveling waves in cellular neural networks, Internat. J. Bifurcat. Chaos 9 (1999) 1307–1319.

[8] C.-H. Hsu, S.-Y. Yang, Structure of traveling waves in delayed cellular neural networks, preprint, 2002.

[9] C.-H. Hsu, S.-Y. Yang, Existence of monotonic traveling waves in lattice dynamical systems, preprint, 2003.

[10] C.-H. Hsu, S.-Y. Yang, On camel-like traveling wave solutions in cellular neural networks, J. Differential Equations 196 (2004) 481–514.

[11] H. Hudson, B. Zinner, Existence of traveling waves for a generalized discrete Fisher's equations, Comm. Appl. Nonlinear Anal. 1 (1994) 23–46.

[12] M. Itoh, P. Julián, L.O. Chua, RTD-based cellular neural networks with multiple steady states, Internat. J. Bifurcat. Chaos 11 (2001) 2913–2959.

[13] J. Juang, S.-S. Lin, Cellular neural networks: mosaic pattern and spatial chaos, SIAM J. Appl. Math. 60 (2000) 891–915.

[14] J.P. Keener, Propagation and its failure in coupled systems of discrete excitable cells, SIAM J. Appl. Math. 47 (1987) 556–572.

[15] J. Mallet-Paret, The Fredholm alternative for functional differential equations of mixed type, J. Dyn. Differential Equations 11 (1999) 1–48.

[16] J. Mallet-Paret, The global structure of traveling waves in spatial discrete dynamical systems, J. Dyn. Differential Equations 11 (1999) 49–127.

[17] P. Weng, J. Wu, Deformation of traveling waves in delayed cellular neural networks, Internat. J. Bifurcat. and Chaos 13 (2003) 797–813.

[18] J. Wu, X. Zou, Asymptotical and periodic boundary value problems of mixed FDEs and wave solutions of lattice differential equations, J. Differential Equations 135 (1997) 315–357.

[19] B. Zinner, Existence of traveling wavefront solutions for discrete Nagumo equation, J. Differential Equations 96 (1992) 1–27.

[20] B. Zinner, G. Harris, W. Hudson, Traveling wavefronts for the discrete Fisher's equation, J. Differential Equations 105 (1993) 46–62.

[21] X. Zou, J. Wu, Local existence and stability of periodic traveling waves of lattice differential equations, Canad. Appl. Math. Quarterly 6 (1998) 397–418.